ALGEBRAS OF FRACTIONS AND STRICT POSITIVSTELLENSÄTZE FOR $*$-ALGEBRAS

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Abstract. In this paper we investigate a $*$-algebra $\mathfrak{X}$ of fractions associated with a unital complex $*$-algebra $\mathcal{A}$. The algebra $\mathfrak{X}$ and its Hilbert space representations are used to prove abstract noncommutative strict Positivstellensätze for $\mathcal{A}$. Multi-grading of $\mathcal{A}$ are studied as technical tools to verify the assumptions of this theorem.

As applications we obtain new strict Positivstellensätze for the Weyl algebra and for the Lie algebra $\mathfrak{g}$ of the affine group of the real line. We characterize integrable representations of the Lie algebra $\mathfrak{g}$ in terms of resolvents of the generators and derive a new integrability criterion for representations of $\mathfrak{g}$.

1. Introduction

Positivstellensätze are fundamental results of real algebraic geometry [PD], [M1]. They represent positive or nonnegative polynomials on semi-algebraic sets in terms of weighted sums of squares of polynomials. Noncommutative strict Positivstellensätze have been proved for the Weyl algebra in [S3] (see also [C]) and for the enveloping algebra of a finite dimensional Lie algebra in [S4]. The technical ingredients of these proofs are Hilbert space representations of certain algebras of fractions. Results of this kind can be considered as steps towards a noncommutative real algebraic geometry (see e.g. [S5] and [HP] for recent surveys).

In the present paper we investigate a fraction $*$-algebra $\mathfrak{X}$ associated with a unital $*$-algebra $\mathcal{A}$. Our main aim is to develop a general method and technical tools for proving noncommutative strict Positivstellensätze of $\mathcal{A}$ by means of the $*$-algebra $\mathfrak{X}$.

Throughout $\mathcal{A}$ is a complex unital $*$-algebra which has no zero-divisors and $\mathcal{S}_O$ is a $*$-invariant left Ore set of $\mathcal{A}$. Further, $\mathcal{S}$ is a unital $*$-invariant countable submonoid of $\mathcal{S}_O$ and $\mathfrak{X}$ is a unital $*$-subalgebra of the fraction $*$-algebra $\mathcal{A}\mathcal{S}_O^{-1}$ such that $\mathcal{A} \subseteq \mathfrak{X}\mathcal{S}$, $\mathfrak{X} \subseteq \mathcal{A}\mathcal{S}^{-1}$ and $\mathcal{S}^{-1}$ is a right Ore subset of $\mathfrak{X}$. Let $\mathcal{S}_G$ denote a $*$-invariant set of generators of $\mathcal{S}$ and $\mathfrak{X}_s$ the quotient $*$-algebra of $\mathfrak{X}$ by the two-sided $*$-ideal generated by $s \in \mathcal{S}$.

Let us explain the contents of the paper. In Section 2 we show how a bounded $*$-representation ρ of $\mathfrak{X}$ satisfying $\ker \rho(s^{-1}) = \{0\}$ for $s \in \mathcal{S}$ gives rise to an (unbounded) $*$-representation π_ρ of the $*$-algebra $\mathcal{A}$. Despite of being essential for the results in Section 3, this construction seems to be useful in unbounded representation theory of $*$-algebras. Representations of the form π_ρ are candidates for the definition of "well-behaved" unbounded representations of the $*$-algebra $\mathcal{A}$ (see also Remark 2 below). Section 3 contains three variants of an abstract strict Positivstellensatz for the $*$-algebra $\mathcal{A}$. Our main abstract strict Positivstellensatz (Theorem 3) can be stated as follows. Assume that the $*$-algebra $\mathfrak{X}$ is algebraically bounded and the inner automorphisms $\alpha_s(\cdot) = s{\cdot}s^{-1}$, $s \in \mathcal{S}$, leave $\mathfrak{X}$ invariant. Let c be a hermitian element of $\mathcal{A}$ and $t \in \mathcal{S}$ such that $t^{-1}c(t^*)^{-1}$ is in $\mathfrak{X}$. If the operators $\pi_\rho(c)$ and $\rho_s(t^{-1}c(t^*)^{-1})$ are strictly positive for all irreducible $*$-representations π_ρ of $\mathcal{A}$ and ρ_s of $\mathfrak{X}_s$ for $s \in \mathcal{S}_G$, then there exists an element $s \in \mathcal{S}_O$ such that scs^* is a finite sum of hermitian squares in the $*$-algebra $\mathcal{A}$. The fraction algebras and the denominator sets used in [S3] and [S4] satisfy the assumptions of Theorem 3. In general it might be not easy to prove that these assumptions are fulfilled. In Section 4 we study multi-graded $*$-algebras and develop some conditions and results which are useful tools to verify the assumptions of Theorem 3.

The second group of results of this paper are two strict Positivstellensätze proved in Sections 5 and 7. The first one (Theorem 5) is about the Weyl algebra $\mathcal{W}(1)$ with denominator set $\mathcal{S}_O{=}\mathcal{S}$

Key words and phrases. Positivstellensatz, fractions, Weyl algebra, enveloping algebra, integrable representations.

generated by $\mathcal{S}_G = \{p \pm \alpha i, q \pm \beta i\}$, where α and β are fixed nonzero reals. The proof uses a result of Kato [K2] about the integrability of the canonical commutation relation. The second application (Theorem 8) concerns the enveloping algebra $\mathcal{E}(\mathfrak{g})$ of the Lie algebra $\mathfrak{g}$ of the $ax+b$-group. Here the denominator set $\mathcal{S}_O = \mathcal{S}$ is generated by $\mathcal{S}_G = \{i\mathsf{a} \pm (\alpha+n)i, i\mathsf{b} \pm \beta i;\ n \in \mathbb{Z}\}$, where α and β are reals such that $\alpha < -1$, $\beta \neq 0$ and α is not an integer and $\{\mathsf{a}, \mathsf{b}\}$ is a basis of $\mathfrak{g}$ satisfying the Lie relation $[\mathsf{a}, \mathsf{b}] = \mathsf{b}$. The results of Section 6 are essentially used in the proof of the Positivstellensatz in Section 7, but they are also of interest in itself. Section 6 contains a description of integrable representations of the Lie algeba $\mathfrak{g}$ in terms of a fraction algebra (Proposition 6 and Theorem 6) and a new integrability criterion (Theorem 7) which is the counterpart of Kato's theorem for representations of the Lie algebra $\mathfrak{g}$.

We close this introduction by collecting some terminology on $*$-algebras and unbounded representations (see [S1] for a detailed treatment of this matter). Suppose that $\mathcal{B}$ is a unital $*$-algebra. A *$*$-representation* π of $\mathcal{B}$ on a dense linear subspace $\mathcal{D}(\pi)$ of a Hilbert space $\mathcal{H}(\pi)$ is an algebra homomorphism of $\mathcal{B}$ into the algebra of linear operators mapping $\mathcal{D}(\pi)$ into itself such that $\pi(1)\varphi = \varphi$ and $\langle \pi(b)\varphi, \psi\rangle = \langle \varphi, \pi(b^*)\psi\rangle$ for $\varphi, \psi \in \mathcal{D}(\pi)$ and $b \in \mathcal{B}$. Here $\langle \cdot, \cdot \rangle$ denotes the scalar product of $\mathcal{H}(\pi)$. The *graph topology* t_π is the locally convex topology on $\mathcal{D}(\pi)$ defined by the seminorms $\varphi \to ||\pi(b)\varphi||$, where $b \in \mathcal{B}$. Let $\mathcal{B}_h = \{b \in \mathcal{B} : b^* = b\}$ be the hermitian part of $\mathcal{B}$ and let $\sum \mathcal{B}^2$ be the cone of all finite sums of hermitian squares b^*b, where $b \in \mathcal{B}$. We denote by $\mathcal{B}_b$ the set of all $b \in \mathcal{B}$ for which there exists a positive number λ such that $\lambda \cdot 1 - b^*b \in \sum \mathcal{B}^2$. Then $\mathcal{B}_b$ is a $*$-algebra [V], see e.g. [S3]. We say that $\mathcal{B}$ is *algebraically bounded* when $\mathcal{B} = \mathcal{B}_b$. We write $T > 0$ for a symmetric operator T on a Hilbert space when $\langle T\psi, \psi\rangle > 0$ for all nonzero vectors ψ in its domain $\mathcal{D}(T)$.

2. Some Algebraic Preliminaries

First let us fix the algebraic setup used throughout this paper. We assume that $\mathcal{S}_O$ is a $*$-invariant left Ore set of $\mathcal{A}$. This means that $\mathcal{S}_O$ is a unital $*$-invariant submonoid of $\mathcal{A}\backslash\{0\}$ (that is, $1 \in \mathcal{S}_O$, $s^* \in \mathcal{S}_O$ and $st \in \mathcal{S}_O$ for $s, t \in \mathcal{S}_O$) satisfying the left Ore condition (that is, for each $s \in \mathcal{S}_O$ and $a \in \mathcal{A}$ there exist $t \in \mathcal{S}_O$ and $b \in \mathcal{A}$ such that $ta = bs$). The symbol 1 always denotes the unit element of $\mathcal{A}$. The $*$-invariance and the left Ore condition imply that $\mathcal{S}_O$ satisfies the right Ore condition (that is, for any $s \in \mathcal{S}_O$ and $a \in \mathcal{A}$ there are $t \in \mathcal{S}_O$ and $b \in \mathcal{A}$ such that $at = sb$). Let $\mathcal{A}\mathcal{S}_O^{-1}$ be the fraction $*$-algebra with denominator set $\mathcal{S}_O$ (see e.g. [R], [GW]). We denote by $\mathcal{S}$ a unital $*$-invariant submonoid of $\mathcal{S}_O$ generated by a countable subset $\mathcal{S}_g$, by $\mathcal{S}_G$ the set $\mathcal{S}_g \cup \mathcal{S}_g^*$ and by $\mathcal{A}_G$ a $*$-invariant set of generators of the algebra $\mathcal{A}$.

Throughout we suppose that $\mathfrak{X}$ is a $*$-subalgebra of $\mathcal{A}\mathcal{S}_O^{-1}$ such that $\mathcal{S}^{-1} \subseteq \mathfrak{X}$ and $\mathcal{A}_G \subseteq \mathfrak{X}\mathcal{S}$. Let $\mathfrak{X}_G$ be a fixed $*$-invariant sets of algebra generators of $\mathfrak{X}$. For $s \in \mathcal{S}$ let $\mathfrak{I}_s$ be the two-sided $*$-ideal of $\mathfrak{X}$ generated by s^{-1} (that is, $\mathfrak{I}_s = \mathfrak{X}s^{-1}\mathfrak{X} + \mathfrak{X}(s^*)^{-1}\mathfrak{X}$) and by $\mathfrak{X}_s = \mathfrak{X}/\mathfrak{I}_s$ the corresponding quotient $*$-algebra. For notational simplicity we denote elements of $\mathfrak{X}$ and their images in $\mathfrak{X}_s$ under the canonical map by the same symbol.

The main assumption used in this paper is the following condition:

(O) $\mathcal{S}^{-1}$ is a right Ore set of the algebra $\mathfrak{X}$, that is, for $s \in \mathcal{S}$ and $x \in \mathfrak{X}$ there exist elements $t \in \mathcal{S}$ and $y \in \mathfrak{X}$ such that $xt^{-1} = s^{-1}y$ (or equivalently $sx = yt$).

The next lemma is often used in what follows. It reformulates the well-known fact ([GW], Lemma 4.21(a)) that finitely many fractions can be brought to a common denominator.

Lemma 1. *Assume that (O) is satisfied. Let $\mathcal{F}$ be a finite subset of $\mathcal{S}$. There exists an element $t_0 \in \mathcal{S}$ such that $st^{-1} \in \mathfrak{X}$ and $t^{-1}s \in \mathfrak{X}$ for all $s \in \mathcal{F}$, where $t = t_0^*t_0$.*

Proof. We first prove by induction on the cardinality that for each finite set $\mathcal{F} \subseteq \mathcal{S}$ there exists $t_1 \in \mathcal{S}$ such that $st_1^{-1} \in \mathfrak{X}$ for all $s \in \mathcal{F}$. Suppose this is true for $\mathcal{F}$. Let $s_1 \in \mathcal{S}$. Since $s_1^{-1} \in \mathfrak{X}$, by assumption (O) there are elements $t_2 \in \mathcal{S}$ and $y \in \mathfrak{X}$ such that $s_1^{-1}t_2^{-1} = t_1^{-1}y$. Then we have $s(t_2s_1)^{-1} = (st_1^{-1})y \in \mathfrak{X}$ for $s \in \mathcal{F}$ and $s_1(t_2s_1)^{-1} = t_2^{-1} \in \mathfrak{X}$ which proves our claim for $\mathcal{F} \cup \{s_1\}$.

Now let $\mathcal{F}$ be a finite subset of $\mathcal{S}$. Applying the statement proved in the preceding paragraph to the set $\mathcal{F}\cup\mathcal{F}^*$, there exists $t_0\in\mathcal{S}$ such that $st_0^{-1}\in\mathfrak{X}$ and $s^*t_0^{-1}\in\mathfrak{X}$. Then we have $s(t_0^*t_0)^{-1}=(st_0^{-1})(t_0{}^*)^{-1}\in\mathfrak{X}$ and $(t_0^*t_0)^{-1}s=((s^*t_0^{-1})(t_0{}^*)^{-1})^*\in\mathfrak{X}$ for $s\in\mathcal{F}$. □

Let $\mathfrak{X}\mathcal{S}=\{xs; x\in\mathfrak{X}, s\in S\}$ and $\mathcal{S}\mathfrak{X}=\{sx; x\in\mathfrak{X}, s\in\mathcal{S}\}$ considered as subsets of $\mathcal{A}\mathcal{S}_O^{-1}$. The next lemma collects some equivalent formulations of condition (O). We omit the details of the simple proofs. In the proof of the implication $(iv)\to(v)$ we use Lemma 1 in order to show that $\mathfrak{X}\mathcal{S}$ is closed under addition.

Lemma 2. *The following are equivalent:*
(i) Condition (O) is satisfied.
(ii) $\mathfrak{X}\mathcal{S}=\mathcal{S}\mathfrak{X}$.
(iii) $\mathfrak{X}\mathcal{S}$ *is $*$-invariant.*
(iv) $\mathfrak{X}\mathcal{S}$ *is closed under multiplication.*
(v) $\mathfrak{X}\mathcal{S}$ *is a $*$-subalgebra of* $\mathcal{A}\mathcal{S}_O^{-1}$.

Suppose that (O) holds. Because $\mathcal{S}^{-1}$ is $*$-invariant and a right Ore set of $\mathfrak{X}$ by (O), it is also a left Ore set and the $*$-algebra $\mathfrak{X}(\mathcal{S}^{-1})^{-1}$ of quotients with denominator set $\mathcal{S}^{-1}$ exists. Since $\mathfrak{X}$ is a $*$-subalgebra of $\mathcal{A}\mathcal{S}_O^{-1}$, it follows from the universal property of algebras of quotients that $\mathfrak{X}(\mathcal{S}^{-1})^{-1}$ is $*$-isomorphic to the $*$-subalgebra $\mathfrak{X}\mathcal{S}$ (by Lemma 2) of $\mathcal{A}\mathcal{S}_O^{-1}$. As assumed above the $*$-algebra $\mathfrak{X}\mathcal{S}$ contains the generator set $\mathcal{A}_G$ of the algebra $\mathcal{A}$. Therefore, we have

$$\mathcal{A}\subseteq\mathfrak{X}\mathcal{S}. \tag{1}$$

The following three conditions are on sets of generators of $\mathcal{S}$ and $\mathfrak{X}$. Because of Lemma 3 below they are convenient tools for the verification of condition (O).

(IA) *For $s\in\mathcal{S}_G$ and $x\in\mathfrak{X}_G$ there is an element $y\in\mathfrak{X}$ such that $xs^{-1}=s^{-1}y$.*

$(A1)$ *For $s\in\mathcal{S}_G$ and $x\in\mathfrak{X}_G$ there exist elements $t\in\mathcal{S}_G$ and $y\in\mathfrak{X}$ such that $xt^{-1}=s^{-1}y$.*

$(A2)$ *Given $s_1,s_2\in\mathcal{S}_G$, there exists an element $t\in\mathcal{S}$ such that $s_1t^{-1}\in\mathfrak{X}$ and $s_2t^{-1}\in\mathfrak{X}$.*

Note that (IA) is a strengthening of $(A1)$. An equivalent formulation of (IA) is that for each generator $s\in\mathcal{S}_G$ (and hence for all $x\in\mathcal{S}$) the inner automorphism $\alpha_s(x):=sxs^{-1}$ of the algebra $\mathcal{A}\mathcal{S}_O^{-1}$ leaves $\mathfrak{X}$ invariant.

Lemma 3. *(i) If $(A1)$ and $(A2)$ are satisfied, then (O) holds.*
(ii) If (IA) is fulfilled, then $(A1)$, $(A2)$ and hence (O) are valid.

Proof. (i): Let $\mathcal{Y}$ denote the set of elements $x\in\mathfrak{X}$ such that for each $s\in\mathcal{S}_G$ there exist $t\in\mathcal{S}_G$ (!) and $y\in\mathfrak{X}$ satisfying $sx=yt$. Let $x_1,x_2\in\mathcal{Y}$ and $s\in\mathcal{S}_G$. Then there are $t_1,t_2\in\mathcal{S}_G$ and $y_1,y_2\in\mathfrak{X}$ such $sx_1=y_1t_1$ and $sx_2=y_2t_2$. Since $t_1\in\mathcal{S}_G$ and $x_2\in\mathcal{Y}$, there exist $t_3\in\mathcal{S}_G$ and $y_3\in\mathfrak{X}$ such that $t_1x_2=y_3t_3$. Then we have $sx_1x_2=y_1t_1x_2=y_1y_3t_3$, so that $x_1x_2\in\mathcal{Y}$. Because $\mathcal{Y}$ contains the set $\mathfrak{X}_G$ of algebra generators by $(A1)$, it follows that $\mathrm{Lin}\,\mathcal{Y}=\mathfrak{X}$.

Since $t_1,t_2\in\mathcal{S}_G$, condition (A2) applies and there exists $t\in\mathcal{S}$ such that $t_1t^{-1},t_2t^{-1}\in\mathfrak{X}$. Then we have $s(\lambda_1x_1+\lambda_2x_2)=(\lambda_1y_1t_1t^{-1}+\lambda_2y_2t_2t^{-1})t\in\mathfrak{X}\cdot t$ for $\lambda_1,\lambda_2\in\mathbb{C}$. This proves that (O) is valid for generators $s\in\mathcal{S}_G$ and for all $x\in\mathrm{Lin}\,\mathcal{Y}=\mathfrak{X}$.

Now suppose s_1 and s_2 are elements of $\mathcal{S}$ such that the assertion of (O) holds for all elements of $\mathfrak{X}$. Therefore, if $x\in\mathcal{S}$, then there are $t_1,t_2\in\mathcal{S}$ and $y_1,y_2\in\mathfrak{X}$ such that $s_1x=y_1t_1$ and $s_2y_1=y_2t_2$. Then, $s_2s_1x=s_2y_1t_1=y_2t_2t_1$, that is, (O) holds for the product s_1s_2 and all $x\in\mathfrak{X}$ as well. Hence condition (O) is valid for arbitrary elements $s\in\mathcal{S}$ and $x\in\mathfrak{X}$.

(ii): Trivially, (IA) implies $(A1)$. Let $s_1,s_2\in\mathcal{S}_G$. Putting $t=s_1s_2$, we have $s_1t^{-1}=\alpha_{s_1}(s_2^{-1})\in\mathfrak{X}$ as follows from $(A3)$ and $s_2t^{-1}=s_1^{-1}\in\mathfrak{X}$. This proves $(A2)$. □

Throughout the rest of this paper we assume that assumption (O) is satisfied.

Now let ρ be a $*$-representation of $\mathfrak{X}$. Since ρ is a right $\mathfrak{X}$-module and $\mathcal{S}^{-1}$ is a right Ore set,

$$\mathcal{D}_{tor}(\rho):=\{\varphi\in\mathcal{D}(\rho): \text{There exists } s\in\mathcal{S} \text{ such that } \rho(s^{-1})\varphi=0\}$$

is a linear subspace of $\mathcal{D}(\rho)$ which is invariant under ρ ([GW], Lemma 4.12). Hence the restriction ρ_{tor} of ρ to the t_ρ-closure of $\mathcal{D}_{tor}(\rho)$ in $\mathcal{D}(\rho)$ is a $*$-representation ρ_{tor} of $\mathfrak{X}$ called the

$\mathcal{S}^{-1}$*-torsion subrepresentation* of ρ. We say that ρ is $\mathcal{S}^{-1}$*-torsion* if $\mathcal{D}_{tor}(\rho) = \mathcal{D}(\rho)$ and that ρ is $\mathcal{S}^{-1}$*-torsionfree* if $\mathcal{D}_{tor}(\rho) = \{0\}$. We shall omit the prefix $\mathcal{S}^{-1}$ if no confusion can arise.

Suppose now that ρ is a *bounded* $*$-representation of $\mathfrak{X}$ on a Hilbert space $\mathcal{H}(\rho) = \mathcal{D}(\rho)$. Then $\mathcal{D}(\rho_{tor})$ is closed subspace of $\mathcal{H}(\rho)$ and ρ is a direct sum of the *torsion subrepresentation* ρ_{tor} on the Hilbert space $\mathcal{D}(\rho_{tor})$ and a *torsionfree subrepresentation* ρ_{tfr} on $\mathcal{D}(\rho_{tfr}) := \mathcal{H}(\rho) \ominus \mathcal{D}(\rho_{tor})$.

Lemma 4. *Suppose that condition (IA) is satisfied. Then each bounded $*$-representation ρ of $\mathfrak{X}$ on a Hilbert space $\mathcal{H}(\rho) = \mathcal{D}(\rho)$ decomposes into a direct sum $\rho = \rho_{tfr} \oplus (\oplus_{s \in \mathcal{S}_G} \rho_s)$ of $*$-representations ρ_{tfr} and ρ_s of $\mathfrak{X}$ such that ρ_{tfr} is torsionfree and $\rho_s(s^{-1}) = 0$ for $s \in \mathcal{S}_G$, so ρ_s factors to a $*$-representation of the $*$-algebra $\mathfrak{X}_s = \mathfrak{X}/\mathfrak{I}_s$.*

Proof. We enumerate the countable set $\mathcal{S}_G$ as $\mathcal{S}_G = \{r_j; j \in N\}$, where either $N = \{1, \cdots, m\}$ with $m \in \mathbb{N}$ or $N = \mathbb{N}$. Put $\mathcal{H}_{r_1} := \ker\ \rho(r_1^{-1})$. Let $x \in \mathfrak{X}$. Since $\mathcal{S}_G$ is $*$-invariant, $r_1^* \in \mathcal{S}_G$ and hence $y := r_1^* x^* (r_1^*)^{-1} \in \mathfrak{X}$ by (IA), so that $\rho(r_1^{-1})\rho(x)\varphi = \rho(y^*)\rho(r_1^{-1})\varphi = 0$ for $\varphi \in \mathcal{H}_{r_1}$. That is, the (bounded) $*$-representation ρ leaves $\mathcal{H}_{r_1}$ invariant. Let ρ_{r_1} and $\tilde{\rho}$ denote the restrictions of ρ to $\mathcal{H}_{r_1}$ and $\mathcal{H}(\rho) \ominus \mathcal{H}_{r_1}$, respectively. Then we have $\rho_{r_1}(r_1^{-1}) = 0$ and $\ker\ \tilde{\rho}(r_1^{-1}) = \{0\}$ by definition. Proceeding in similar manner by induction, we obtain an orthogonal direct sum of $*$-representations ρ_{r_j} of $\mathfrak{X}$ on subspaces $\mathcal{H}_{r_j}$, $j \in N$. Clearly, for the restriction ρ_{tfr} of ρ to the invariant subspace $\mathcal{H}_0 := \mathcal{H}(\rho) \ominus (\oplus_{j \in N} \mathcal{H}_{r_j})$ we have $\ker \rho(s^{-1}) = \{0\}$ for $s \in \mathcal{S}_G$ and hence for all $s \in \mathcal{S}$. This means that ρ_{tfr} is torsionfree. ◻

Let us illustrate the preceding decomposition by a very simple example.

Example 1. Let $\mathcal{A} = \mathbb{C}[x]$ be the $*$-algebra of polynomials in a hermitian variable x. Set $\mathcal{S}_G = \{s{:=}x^2{+}1\}$ and $\mathcal{S} = \mathcal{S}_O = \{s^n; n \in \mathbb{N}_0\}$. Let $\mathfrak{X}$ be the unital $*$-subalgebra of $\mathcal{A}\mathcal{S}_O^{-1}$ generated by $a{:=}s^{-1}$ and $b{:=}xs^{-1}$. It is not difficult to show that each $*$-representation ρ of $\mathfrak{X}$ is of the form

$$\rho(p(a,b)) = \int_{\mathcal{C}} p(\lambda, \mu) dE(\lambda, \mu), \quad p \in \mathbb{C}[a,b],$$

for some spectral measure E on $\mathcal{H}(\rho)$ supported on the circle $\mathcal{C}$ given by the equation $\lambda^2 + \mu^2 = \lambda$. Then we have $\mathcal{D}(\rho_{tor}) = \mathcal{D}(\rho_s) = E((0,0))\mathcal{H}(\rho)$, $\mathcal{D}(\rho_{tfr}) = E(\mathcal{C} \backslash (0,0))\mathcal{H}(\rho)$ and $\rho_s(p(a,b))\varphi = p(0,0)\varphi$ for $\varphi \in E((0,0))\mathcal{H}(\rho)$. Note that $b^2 \in \mathcal{J}_s$ and $b \notin \mathcal{J}_s$, but $\rho_s(b) = 0$ (see e.g. Lemmas 8 and 9 below).

3. Representations of $\mathcal{A}$ Associated with Torsionfree Representations of $\mathfrak{X}$

Suppose that ρ is a *bounded* $\mathcal{S}^{-1}$*-torsionfree* $*$-representation of the $*$-algebra $\mathfrak{X}$ on a Hilbert space $\mathcal{D}(\rho) = \mathcal{H}$. That ρ is torsionfree means that $\ker \rho(s^{-1}) = \{0\}$ for all $s \in \mathcal{S}$. Our aim is to associate an (unbounded) $*$-representation π_ρ of the $*$-algebra $\mathcal{A}$ with ρ. Define

$$\mathcal{D}_\rho = \cap_{s \in \mathcal{S}}\ \rho(s^{-1})\mathcal{H}. \qquad (2)$$

Lemma 5. (i) *$\mathcal{D}_\rho$ is dense in the Hilbert space $\mathcal{H}$.*
(ii) $\rho(x)\mathcal{D}_\rho \subseteq \mathcal{D}_\rho$ for $x \in \mathfrak{X}$.
(iii) $\rho(s^{-1})\mathcal{D}_\rho = \mathcal{D}_\rho$ for $s \in \mathcal{S}$.

Proof. (i): The main technical tool for proving this assertion is the so-called *Mittag-Leffler lemma* (see e.g. [S1], p. 15). Let us develop the necessary setup for this result.

We enumerate the countable set $\mathcal{S}_G$ of generators as $\mathcal{S}_G = \{r_j; j \in N\}$ such that $r_1 = 1$, where $N = \{1, \cdots, m\}$ with $m \in \mathbb{N}$ or $N = \mathbb{N}$. For $n \in \mathbb{N}$ let $\mathcal{S}^n$ denote the set of all products $r_{j_1} \ldots r_{j_n}$, where $j_1 \leq n, \ldots, j_r \leq n$ and $j_1, \ldots, j_n \in N$. Since the set $\mathcal{S}^n$ is finite, it follows from Lemma 1 that for each $n \in \mathbb{N}$ there exists an element $t_n = t_n^* \in \mathcal{S}$ such that $st_n^{-1} \in \mathfrak{X}$ for all $s \in \mathcal{S}^n$ and $t_n t_{n+1}^{-1} \in \mathfrak{X}$. Setting $\mathcal{S}^0 = \{1\}$ and $t_0 = 1$, the latter is also satisfied for $n = 0$.

For $n \in \mathbb{N}_0$, let E_n denote the vector space $\rho(t_n^{-1})\mathcal{H}$ equipped with the scalar product defined by $(\varphi, \psi)_n = \langle \rho(t_n^{-1})^{-1}\varphi, \rho(t_n^{-1})^{-1}\psi \rangle$, where $\varphi, \psi \in E_n$. Since E_n is the range of the bounded injective operator $\rho(t_n^{-1})$, $(E_n, (\cdot,\cdot)_n)$ is a Hilbert space with norm $||\varphi||_n = ||\rho(t_n^{-1})^{-1}\varphi||$.

We first show that E_{n+1} is a subspace of E_n and that $||\cdot||_n \leq c_n ||\cdot||_{n+1}$ for some positive constant c_n. For let $\psi \in \mathcal{H}$ and set $\varphi := \rho(t_{n+1}^{-1})\psi$. Since $t_n t_{n+1}^{-1} \in \mathfrak{X}$ by the choice of elements

t_k , we obtain $\varphi = \rho(t_{n+1}^{-1})\psi = \rho(t_n^{-1})\rho(t_nt_{n+1}^{-1})\psi$ which proves that $E_{n+1} \subseteq E_n$. By definition we have $||\varphi||_{n+1} = ||\psi||$ and hence $||\varphi||_n = ||\rho(t_nt_{n+1}^{-1})\psi|| \leq ||\rho(t_nt_{n+1}^{-1})||\ ||\varphi||_{n+1}$.

Next we prove that E_{n+1} is dense in the normed space $(E_n, ||\cdot||_n)$. For this it suffices to show that each vector $\zeta \in E_n$ which is orthogonal to E_{n+1} in the Hilbert space $(E_n, (\cdot,\cdot)_n)$ is the null vector. Put $\xi := \rho(t_n^{-1})\zeta$. That the vector ζ is orthogonal to E_{n+1} means that

$$\begin{aligned}0 &= (\zeta, \rho(t_{n+1}^{-1})\varphi)_n = \langle \rho(t_n^{-1})^{-1}\zeta, \rho(t_n^{-1})^{-1}\rho(t_{n+1}^{-1})\varphi\rangle = \langle \xi, \rho(t_n^{-1})^{-1}\rho(t_n^{-1})\rho(t_nt_{n+1}^{-1})\varphi\rangle \\ &= \langle \xi, \rho(t_nt_{n+1}^{-1})\varphi\rangle = \langle \rho(t_nt_{n+1}^{-1})^*\xi, \varphi\rangle = \langle \rho(t_{n+1}^{-1}t_n)\xi, \varphi\rangle = \langle \rho(t_{n+1}^{-1}t_n)\rho(t_n^{-1})\zeta, \varphi\rangle = \langle \rho(t_{n+1}^{-1})\zeta, \varphi\rangle\end{aligned}$$

for all $\varphi \in \mathcal{H}$, where we freely used the properties of the $*$-representation ρ of $\mathfrak{X}$ and of the larger $*$-algebra $\mathcal{A}\mathcal{S}_O^{-1}$. Thus we obtain $\rho(t_{n+1}^{-1})\zeta = 0$. Since ρ is torsionfree, $\ker \rho(t_{n+1}^{-1}) = \{0\}$. Hence we get $\zeta = 0$. This proves that E_{n+1} is dense in E_n.

In the preceding two paragraphs we have shown that the assumptions of the Mittag-Leffler lemma (see [S1], Lemma 1.1.2) are fulfilled. From this result it follows that the vector space $E_\infty := \cap_{n\in\mathbb{N}_0} E_n$ is dense in the normed space $E_0 = \mathcal{H}$. Obviously, $\mathcal{D}_\rho \subseteq E_\infty$. Let $s \in \mathcal{S}$. Then $s \in \mathcal{S}^n$ for some $n \in \mathbb{N}$ and hence $\rho(t_n^{-1})\mathcal{H} = \rho(s^{-1})\rho(st_n^{-1})\mathcal{H} \subseteq \rho(s^{-1})\mathcal{H}$. This in turn yields $E_\infty \subseteq \mathcal{D}_\rho$. Therefore, $\mathcal{D}_\rho = E_\infty$ is dense in $\mathcal{H}$.

(ii): Suppose that $\varphi \in \mathcal{D}_\rho$ and $x \in \mathfrak{X}$. Let $s \in \mathcal{S}$. By assumption (O) there exist elements $t \in \mathcal{S}$ and $y \in \mathfrak{X}$ such that $sx = yt$, so that $xt^{-1} = s^{-1}y$. From the definition (2) of $\mathcal{D}_\rho$, there is a vector $\psi \in \mathcal{H}$ such that $\varphi = \rho(t^{-1})\psi$. Then we have $\rho(x)\varphi = \rho(xt^{-1})\psi = \rho(s^{-1})\rho(y)\psi \in \rho(s^{-1})\mathcal{H}$. Since $s \in \mathcal{S}$ was arbitrary, we have shown that $\rho(x)\varphi \in \cap_{s\in\mathcal{S}}\ \rho(s^{-1})\mathcal{H} = \mathcal{D}_\rho$.

(iii): Suppose $s \in \mathcal{S}$. Since $\rho(s^{-1})\mathcal{D}_\rho \subseteq \mathcal{D}_\rho$ by (ii), it suffices to show that each vector $\varphi \in \mathcal{D}_\rho$ belongs to $\rho(s^{-1})\mathcal{D}_\rho$. According to the definition of $\mathcal{D}_\rho$, we have $\varphi \in \rho(s^{-1})\mathcal{H}$ and $\varphi \in \rho((ts)^{-1})\mathcal{H}$ for each $t \in \mathcal{S}$, that is, there are vectors $\psi \in \mathcal{H}$ and $\eta_t \in \mathcal{H}$ such that $\varphi = \rho(s^{-1})\psi = \rho((ts)^{-1})\eta_t$. Since $\ker \rho(s^{-1}) = \{0\}$, the latter implies that $\psi = \rho(t^{-1})\eta_t$, so that $\psi \in \cap_{t\in\mathcal{S}}\ \rho(t^{-1})\mathcal{H} = \mathcal{D}_\rho$ and $\varphi = \rho(s^{-1})\psi$. □

Let $a \in \mathcal{A}$. Suppose that s is an element of $\mathcal{S}$ such that $as^{-1} \in \mathfrak{X}$. From (1) it follows that such an element s always exists. Define

$$\pi_\rho(a)\varphi := \rho(as^{-1})\rho(s^{-1})^{-1}\varphi, \quad \varphi \in \mathcal{D}_\rho. \tag{3}$$

Theorem 1. *Let ρ be a bounded $\mathcal{S}^{-1}$-torsionfree $*$-representation of the $*$-algebra $\mathfrak{X}$ on a Hilbert space $\mathcal{H}$. Then π_ρ is a well-defined closed $*$-representation of the $*$-algebra $\mathcal{A}$ with Frechet graph topology on the dense domain $\mathcal{D}(\pi_\rho) := \mathcal{D}_\rho$ of the Hilbert space $\mathcal{H}$. For $s \in \mathcal{S}$ and $\varphi \in \mathcal{D}(\pi_\rho)$ we have $\pi_\rho(s)\mathcal{D}(\pi_\rho) = \mathcal{D}(\pi_\rho)$ and $\pi_\rho(s)\varphi = \rho(s^{-1})^{-1}\varphi$. The $*$-representation π_ρ of $\mathcal{A}$ is irreducible if and only if the $*$-representation ρ of $\mathfrak{X}$ is irreducible.*

Proof. We first show that the operator $\pi_\rho(a)$ is well-defined, that is, $\pi_\rho(a)$ (3) does not depend on the particular element s of $\mathcal{S}$ satisfying $as^{-1} \in \mathfrak{X}$. Let $\widetilde{s} \in \mathcal{S}$ be another element such that $a\widetilde{s}^{-1} \in \mathfrak{X}$. By Lemma 1 there exists $t \in \mathcal{S}$ such that $st^{-1} \in \mathfrak{X}$ and $\widetilde{s}t^{-1} \in \mathfrak{X}$. Then $at^{-1} = (as^{-1})(st^{-1}) \in \mathfrak{X}$. Let r denote s or $\widetilde{s}$. Writing $\varphi = \rho(t^{-1})\psi$ with $\psi \in \mathcal{H}$, we compute

$$\begin{aligned}\rho(ar^{-1})\rho(r^{-1})^{-1}\varphi &= \rho(ar^{-1})\rho(r^{-1})^{-1}\rho(t^{-1})\psi = \rho(ar^{-1})\rho(r^{-1})^{-1}\rho(r^{-1}rt^{-1})\psi = \\ &= \rho(ar^{-1})\rho(rt^{-1})\psi = \rho(ar^{-1}rt^{-1})\rho(t^{-1})^{-1}\psi = \rho(at^{-1})\rho(t^{-1})^{-1}\varphi,\end{aligned} \tag{4}$$

so $\rho(as^{-1})\rho(s^{-1})^{-1}\varphi = \rho(a\widetilde{s}^{-1})\rho(\widetilde{s}^{-1})^{-1}\varphi$. This shows that the operator $\pi\rho(a)$ is well-defined.

Since $\rho(s^{-1})^{-1}\varphi \in \mathcal{D}_\rho$ and $\rho(as^{-1})\rho(s^{-1})^{-1}\varphi \in \mathcal{D}_\rho$ by Lemma 5,(ii) and (iii), we have $\pi_\rho(a)\varphi \in \mathcal{D}_\rho$, that is, $\pi_\rho(a)$ maps the domain $\mathcal{D}(\pi_\rho)$ into itself.

Suppose that $a, b \in \mathcal{A}$. We shall prove that $\pi_\rho(a+b) = \pi_\rho(a)+\pi_\rho(b)$ and $\pi_\rho(ab) = \pi_\rho(a)\pi_\rho(b)$.

By (1) there are elements $s_1, s_2 \in \mathcal{S}$ such that $as_1^{-1} \in \mathfrak{X}$ and $bs_2^{-1} \in \mathfrak{X}$. By Lemma 1 we can find $s \in \mathcal{S}$ such that $s_1s^{-1} \in \mathfrak{X}$ and $s_2s^{-1} \in \mathfrak{X}$. Since then $as^{-1} \in \mathfrak{X}$, $bs^{-1} \in \mathfrak{X}$ and $(a+b)s^{-1} \in \mathfrak{X}$, the relation

$$\rho(as^{-1})\rho(s^{-1})^{-1}\varphi + \rho(bs^{-1})\rho(s^{-1})^{-1}\varphi = \rho((a+b)s^{-1})\rho(s^{-1})^{-1}\varphi,\ \varphi \in \mathcal{D}_\rho,$$

says that $\pi_\rho(a+b) = \pi_\rho(a) + \pi_\rho(b)$.

From (1), there exist elements $t_1, t_2, t_3, t_4 \in \mathcal{S}$ such that $at_1^{-1}, bt_2^{-1}, abt_3^{-1} \in \mathfrak{X}$ and $t_1bt_4^{-1} \in \mathfrak{X}$. By Lemma 1 there is an element $t \in \mathcal{S}$ such that $t_jt^{-1} \in \mathfrak{X}$ for $j = 1, 2, 3, 4$. Then we have

$abt^{-1} = (abt_3^{-1})(t_3t^{-1}) \in \mathfrak{X}$, $t_1bt^{-1} = (t_1bt_4^{-1})(t_4t^{-1}) \in \mathfrak{X}$ and $bt^{-1} = (bt_2^{-1})(t_2t^{-1}) \in \mathfrak{X}$. Let $\varphi \in \mathcal{D}_\rho$. Inserting the corresponding definitions of $\pi_\rho(ab), \pi_\rho(a)$ and $\pi_\rho(b)$ we derive

$$\pi_\rho(ab)\varphi = \rho(abt^{-1})\rho(t^{-1})^{-1}\varphi = \rho(at_1^{-1})\rho(t_1bt^{-1})\rho(t^{-1})^{-1}\varphi$$
$$= \rho(at_1^{-1})\rho(t_1^{-1})^{-1}\rho(t_1^{-1})\rho(t_1bt^{-1})\rho(t^{-1})^{-1}\varphi = \pi_\rho(a)\rho(t_1^{-1}t_1bt^{-1})\rho(t^{-1})^{-1}\varphi$$
$$= \pi_\rho(a)\rho(bt^{-1})\rho(t^{-1})^{-1}\varphi = \pi_\rho(a)\pi_\rho(b)\varphi.$$

Finally, we verify that $\langle \pi_\rho(a)\varphi, \psi\rangle = \langle \varphi, \pi_\rho(a^+)\psi\rangle$ for $a \in \mathcal{A}$ and $\varphi, \psi \in \mathcal{D}_\rho$. From (1) and Lemma 1 there are elements $t_1, t_2 \in \mathcal{S}$ and $t{=}t^* \in \mathcal{S}$ such that $at_1^{-1}, a^*t_2^{-1} \in \mathfrak{X}$ and $t_1t^{-1}, t_2t^{-1} \in \mathfrak{X}$. Since then $at^{-1} \in \mathfrak{X}$ and $a^*t^{-1} \in \mathfrak{X}$, using that $\rho(t^{-1})$ is bounded self-adjoint operator we compute

$$\langle \pi_\rho(a)\varphi, \psi\rangle = \langle \rho(at^{-1})\rho(t^{-1})^{-1}\varphi, \psi\rangle = \langle \rho(t^{-1})^{-1}\varphi, \rho((at^{-1})^*)\psi\rangle$$
$$= \langle \rho(t^{-1})^{-1}\varphi, \rho(t^{-1}a^*)\rho(t^{-1})\rho(t^{-1})^{-1}\psi\rangle = \langle \rho(t^{-1})^{-1}\varphi, \rho(t^{-1}a^*t^{-1})\rho(t^{-1})^{-1}\psi\rangle$$
$$= \langle \rho(t^{-1})^{-1}\varphi, \rho(t^{-1})\rho(a^*t^{-1})\rho(t^{-1})^{-1}\psi\rangle = \langle \rho(t^{-1})\rho(t^{-1})^{-1}\varphi, \pi_\rho(a^*)\psi\rangle = \langle \varphi, \pi_\rho(a^*)\psi\rangle,$$

where we used the fact that $\rho(t^{-1})\mathcal{D}_\rho = \mathcal{D}_\rho$ according to Lemma 5(iii).

Clearly, $\pi_\rho(\mathbf{1})\varphi{=}\varphi$ for $\varphi \in \mathcal{D}(\pi)$. Recall from Lemma 5(i) that $\mathcal{D}_\rho$ is dense in $\mathcal{H}$. Thus, we have shown that π_ρ is a $*$-representation of $\mathcal{A}$ on the dense domain $\mathcal{D}_\rho$ of the Hilbert space $\mathcal{H}$.

Let $s \in \mathcal{S}$. Because $ss^{-1}{=}\mathbf{1} \in \mathfrak{X}$, we have $\pi_\rho(s)\varphi = \rho(s^{-1})^{-1}\varphi$ for $\varphi \in \mathcal{D}_\rho$. Since $\rho(s^{-1})\mathcal{D}_\rho = \mathcal{D}_\rho$ by Lemma 5(iii), it follows that $\pi_\rho(s)\mathcal{D}(\pi_\rho) = \mathcal{D}(\pi_\rho)$.

To prove the assertion concerning the graph topology of π_ρ, we retain the notation from the proof of Lemma 5(i). Since $\pi_\rho(t_n)\varphi = \rho(t_n^{-1})^{-1}\varphi$ for $\varphi \in \mathcal{D}_\rho$ and $n \in \mathbb{N}$, the graph seminorm $||\pi_\rho(t_n)\varphi||$ is just the norm $||\varphi||_n$. Let $a \in \mathcal{A}$. Applying once more (1) there is an element $t \in \mathcal{S}$ such that $at^{-1} \in \mathfrak{X}$. We can find a number $n \in \mathbb{N}$ such that $t \in \mathcal{S}^n$. Since then $tt_n^{-1} \in \mathfrak{X}$, we have $at_n^{-1} = (at^{-1})(tt_n^{-1}) \in \mathfrak{X}$ and hence

$$||\pi_\rho(a)\varphi|| = ||\rho(at_n^{-1})\rho(t_n^{-1})^{-1}\varphi|| = ||\rho(at_n^{-1})\pi_\rho(t_n)\varphi|| \le ||\rho(at_n^{-1})||\ ||\varphi||_n.$$

The preceding shows that the graph topology of π_ρ is generated by the family of norms $||\cdot||_n$, $n \in \mathbb{N}$. Hence it is the projective limit topology of the countable family of Hilbert spaces $(E_n, ||\cdot||_n)$ on $E_\infty = \cap_n E_n = \mathcal{D}_\rho$. Therefore, the graph topology of π_ρ is metrizable and complete. The latter implies in particular that the representation π_ρ is closed.

It remains to prove the assertion about the irreducibility. Recall that a $*$-representation π_ρ is irreducible if and only 0 and I are the only projections in the strong commutant $\pi_\rho(\mathcal{A})'_s$ ([S1], 8.3.5). Hence it suffices to show that $\pi_\rho(\mathcal{A})'_s$ is equal to the commutant $\rho(\mathfrak{X})'$. Suppose that $T \in \pi_\rho(\mathcal{A})'_s$. By definition T maps $\mathcal{D}(\pi_\rho)$ into itself and we have $T\pi_\rho(a)\varphi = \pi_\rho(a)T\varphi$ for all $a \in \mathcal{A}$ and $\varphi \in \mathcal{D}(\pi_\rho)$. Let $x \in \mathfrak{X}_G$ and $\psi \in \mathcal{D}(\pi_\rho)$. Then x is of the form $x = as^{-1}$ with $a \in \mathcal{A}$ and $s \in \mathcal{S}$ and $\varphi := \rho(s^{-1})\psi$ belongs to $\mathcal{D}(\pi_\rho)$ by Lemma 5(ii). Applying (3) twice we derive

$$T\rho(x)\psi = T\rho(as^{-1})\rho(s^{-1})^{-1}\varphi = T\pi_\rho(a)\varphi = \pi_\rho(a)T\varphi = \rho(as^{-1})\rho(s^{-1})^{-1}T\varphi$$
$$= \rho(as^{-1})\pi_\rho(s)T\varphi = \rho(as^{-1})T\pi_\rho(s)\varphi = \rho(x)T\rho(s^{-1})^{-1}\varphi = \rho(x)T\psi.$$

Since $\mathcal{D}(\pi_\rho)$ is dense in $\mathcal{H}$, $T\rho(x) = \rho(x)T$. Because $\mathfrak{X}_G$ generates the algebra $\mathfrak{X}$, T is in the commutant $\rho(\mathfrak{X})'$. Conversely, if T is in $\rho(\mathfrak{X})'$, it follows at once from the definitions (2) of $\mathcal{D}(\pi_\rho)$ and (3) of π_ρ that T belongs to $\pi_\rho(\mathcal{A})'_s$. □

Remark 1. The Mittag-Leffler lemma used in the proof of Lemma 5 even states that $E_\infty = \mathcal{D}(\pi_\rho)$ is dense in each Hilbert space $(E_n, ||\cdot||_n)$ for $n \in \mathbb{N}$. This implies that $\mathcal{D}(\pi_\rho)$ is core for each operator $\rho(s^{-1})^{-1}$ for $s \in \mathcal{S}$.

Remark 2. A fundamental problem in unbounded representation theory of $*$-algebras is to select and to classify classes of "well-behaved" $*$-representations among the large variety of representations. An approach to this problem have been proposed in [SS]. Fraction algebras give another possibility by defining well-behaved $*$-representations of the $*$-algebra $\mathcal{A}$ as those of the form π_ρ. Propositions 5 and 7 below support such a definition.

Example 2. Retain the notation of Example 1 and assume that ρ is torsionfree, that is, $E((0,0)) = 0$. Recall that $x = ba^{-1}$ in the $*$-algebra $\mathcal{A}\mathcal{S}_O^{-1}$. For $\varphi \in \mathcal{D}(\pi_\rho) = \cap_{n=1}^{\infty} \rho(a^n)\mathcal{H}$ we have $\pi_\rho(x)\varphi = \rho(b)\rho(a)^{-1}\varphi$ and

$$\pi_\rho(q(x))\varphi = \int_{\mathcal{C}} q(\mu\lambda^{-1})\ dE(\lambda,\mu)\varphi, \quad q \in \mathbb{C}[x].$$

4. Abstract Strict Positivstellensätze

In addition to condition (O) we now essentially use the following assumption:
(AB) *The $*$-algebra $\mathfrak{X}$ is algebraically bounded, that is, for each $x \in \mathfrak{X}$ there is a positive number λ such that $\lambda{\cdot}1 - x^*x \in \sum \mathfrak{X}^2$.*

Note that condition (AB) implies that all $*$-representations of $\mathfrak{X}$ act by bounded operators.
The three theorems in this section are abstract strict Positivstellensätze for the $*$-algebra $\mathcal{A}$.

Theorem 2. *Suppose that conditions (O) and (AB) are satisfied. Let $a \in \mathcal{A}_h$. Suppose there is an element $t \in \mathcal{S}$ such that $t^{-1}a(t^*)^{-1} \in \mathfrak{X}$ and the following assumptions are fulfilled:*
(i) For each irreducible $\mathcal{S}^{-1}$-torsionsfree $*$-representation ρ of $\mathfrak{X}$ on a Hilbert space $\mathcal{H}(\rho) = \mathcal{D}(\rho)$ there exists a bounded self-adjoint operator $T_\rho > 0$ on $\mathcal{H}(\rho)$ such that $\pi_\rho(a) \geq T_\rho$.
(ii) $\rho_{tor}(t^{-1}a(t^*)^{-1}) > 0$ for each irreducible $\mathcal{S}^{-1}$-torsion $*$-representation ρ_{tor} of $\mathfrak{X}$.
Then there exists an element $s \in \mathcal{S}_O$ such that $s^*as \in \sum \mathcal{A}^2$.

The following simple lemma is used in the proofs of Theorems 2 and 4.

Lemma 6. *Suppose that $b \in \mathcal{A}$, $r \in \mathcal{S}$ and $x := r^{-1}b(r^*)^{-1} \in \mathfrak{X}$. Then for any $\mathcal{S}^{-1}$-torsionfree $*$-representation ρ of $\mathfrak{X}$ we have*

$$(5) \qquad \langle \rho(x)\varphi, \varphi\rangle = \langle \pi_\rho(b)\rho((r^*)^{-1})\varphi, \rho((r^*)^{-1})\varphi\rangle, \ \varphi \in \mathcal{D}(\pi_\rho).$$

Proof. By our assumption (O) there are elements $t \in \mathcal{S}$ and $y \in \mathfrak{X}$ such that $rx = yt = b(r^*)^{-1}$. If $\varphi \in \mathcal{D}(\pi_\rho)$, then $\psi := \rho(t^{-1})^{-1}\varphi \in \mathcal{D}(\pi_\rho)$ by Lemma 5(iii). Using these facts we compute

$$\begin{aligned}&\langle \rho(x)\varphi, \varphi\rangle = \langle \rho(r^{-1}yt)\rho(t^{-1})\psi, \varphi\rangle = \langle \rho(r^{-1}y)\psi, \varphi\rangle = \langle \rho(y)\psi, \rho((r^*)^{-1})\varphi\rangle = \\ &\langle \rho(b(tr^*)^{-1})\psi, \rho((r^*)^{-1})\varphi\rangle = \langle \pi_\rho(b)\rho((tr^*)^{-1})\psi, \rho((r^*)^{-1})\varphi\rangle \\ &= \langle \pi_\rho(b)\rho((r^*)^{-1})\rho(t^{-1})\psi, \rho((r^*)^{-1})\varphi\rangle = \langle \pi_\rho(b)\rho((r^*)^{-1})\varphi, \rho((r^*)^{-1})\varphi\rangle\end{aligned}$$

where the fifth equality follows from formula (3), because we have $y = b(tr^*)^{-1} \in \mathfrak{X}$. $\square$

Proof of Theorem 2:
Set $y := t^{-1}a(t^*)^{-1}$. Our first aim is to show that $y \in \sum \mathfrak{X}^2$. The proof of this assertion is based on a now standard separation argument which is has been first used in [S2], see e.g. Sections 5.1 and 5.2 in [S5] for the noncommutative case.

Assume to the contrary that y is not in $\sum \mathfrak{X}^2$. Since $\mathfrak{X}$ is algebraically bounded by assumption (AB), the unit element 1 of $\mathfrak{X}$ is an algebraic inner point of the wedge $\sum\ \mathfrak{X}^2$ of the real vector space $\mathfrak{X}_h$. Therefore, by Eidelheit's separation theorem (see e.g. [J], 0.2.4), there exists an $\mathbb{R}$-linear functional $F \not\equiv 0$ on $\mathfrak{X}_h$ such that $F(y) \leq 0$ and $F(\sum \mathfrak{X}^2) \geq 0$. There is no loss of generality to assume that $F(1) = 1$. By a standard application of the Krein-Milman theorem (see e.g. [J], 0.3.6 and 1.8.3) it follows that this functional F can be choosen to be extremal (that is, if G is another $\mathbb{R}$-linear functional on $\mathfrak{X}_h$ such that $G(y) \leq 0$, $G(1) = 1$ and $F(x) \geq G(x) \geq 0$ for all $x \in \sum \mathfrak{X}^2$, then $G = F$). We extend F to a $\mathbb{C}$-linear functional, denoted also by F, on $\mathfrak{X}$. Then F is an extremal state of the $*$-algebra $\mathfrak{X}$. Let ρ_F be the $*$-representation of $\mathfrak{X}$ which is associated with F by the GNS-construction. Using once more that $\mathfrak{X}$ is algebraically bounded, it follows that all operators $\rho_F(x)$, $x \in \mathfrak{X}$, are bounded, so we can assume that $\mathcal{D}(\rho_F) = \mathcal{H}(\rho_F)$. Since the state F of $\mathfrak{X}$ is extremal, ρ_F is irreducible. Therefore, by the decomposition of ρ_F discussed in Section 2, ρ_F is either an $\mathcal{S}^{-1}$-torsion or an $\mathcal{S}^{-1}$-torsionfree $*$-representation.

The crucial step of this proof is to show that $\rho_F(y) > 0$. If ρ_F is torsion, then we have $\rho_F(y) > 0$ by assumption (ii). Now we suppose that $\rho := \rho_F$ is torsionfree. Combining equation

(5) in Lemma 6, applied with $a = b$, $r = t$, $x = y$, and the assumption $\pi_\rho(a) \geq T_\rho$, we obtain

$$\langle \rho(y)\varphi, \varphi\rangle \geq \langle T_\rho \rho((t^*)^{-1})\varphi, \rho((t^*)^{-1})\varphi\rangle \tag{6}$$

for $\varphi \in \mathcal{D}(\pi_\rho)$. Because $\rho(y)$, T_ρ and $\rho((t^*)^{-1})$ are bounded operators and $\mathcal{D}(\pi_\rho)$ is dense in $\mathcal{H}(\rho)$ by Lemma 5, it follows that equation (6) holds for arbitrary vectors $\varphi \in \mathcal{H}(\rho)$. Since $T_\rho > 0$ and ker $\rho((t^*)^{-1}) = \{0\}$ because $\rho = \rho_F$ is torsionfree, (6) implies that $\rho_F(y) > 0$.

Thus we have $\rho_F(y) > 0$ as just shown and $F(y) \leq 0$ by construction. Since $F \not\equiv 0$, this is the desired contraction. Therefore, $y \in \sum \mathfrak{X}^2$.

We write y as a finite sum $\sum_i y_i^* y_i$ with $y_i \in \mathfrak{X}$. From the Ore property of the set $\mathcal{S}_O$ it follows that for all elements $y_i t^* \in \mathcal{A}\mathcal{S}_O^{-1}$ there is a common right denominator, that is, there exist elements $s \in \mathcal{S}_O$ and $a_i \in \mathcal{A}$ such that $y_i t^* = a_i s^{-1}$ for all i. Then $y = t^{-1}a(t^*)^{-1} = \sum_i y_i^* y_i$ implies that $a = \sum_i (s^*)^{-1} a_i^* a_i s^{-1}$ and so $s^* a s = \sum_i a_i^* a_i \in \sum \mathcal{A}^2$. □

Assuming the stronger condition (IA) instead of (O) we have the following stronger result.

Theorem 3. *Asumme that conditions (IA) and (AB) are satisfied. Let $a \in \mathcal{A}_h$. Suppose there is an element $t \in \mathcal{S}$ such that $t^{-1}a(t^*)^{-1} \in \mathfrak{X}$ and the following assumptions are fulfilled:*
(i) For each irreducible $\mathcal{S}^{-1}$-torsionfree $*$-representation ρ of $\mathfrak{X}$ on a Hilbert space $\mathcal{H}(\rho) = \mathcal{D}(\rho)$ there exists a bounded self-adjoint operator $T_\rho > 0$ on $\mathcal{H}(\rho)$ such that $\pi_\rho(a) \geq T_\rho$.
(ii) $\rho_s(t^{-1}a(t^*)^{-1}) > 0$ for each irreducible $*$-representation ρ_s of the $*$-algebra $\mathfrak{X}_s$ and $s \in \mathcal{S}_G$. Then there exists an element $s \in \mathcal{S}_O$ such that $s^* a s \in \sum \mathcal{A}^2$.

Proof. Since condition (IA) holds by assumption, (O) is satisfied by Lemma 3 and each torsion $*$-representation ρ_{tor} is a direct sum of representations ρ_s of $\mathfrak{X}$ such that $\rho_s(s^{-1}) = 0$ for $s \in \mathcal{S}_G$ by Lemma 4. Therefore, assumption (ii) above implies assumption (ii) of Theorem 2, so the assertion follows from Theorem 2. □

Remark 3. Let a be an element of $\mathcal{A}$ satisfiying assumption (i) of Theorems 2 or 3. By (1) there exists $t \in \mathcal{S}$ such that $t^{-1}a(t^*)^{-1} \in \mathfrak{X}$. Moreover, if $t^{-1}a(t^*)^{-1} \in \mathfrak{X}$ and $s \in \mathcal{S}_G$, then $(st)^{-1}a((st)^*)^{-1} \in \mathcal{J}_s$ and so $\rho_s((st)^{-1}a((st)^*)^{-1}) = 0$ for any $*$-representation ρ_s of $\mathfrak{X}_s$. Hence it is crucial in both theorems to find an element t for which assumption (ii) holds as well.

Remark 4. Let us consider the trivial case when $\mathcal{S} = \{1\}$. Then we have $\mathcal{A} = \mathfrak{X}$ and (O) trivially holds. Hence Theorem 2 gives the following asssertion (see e.g. [S5], Proposition 15) for an *algebraically bounded* $*$-algebra $\mathfrak{X}$: *Let $a \in \mathfrak{X}_h$. If for each irreducible $*$-representation ρ of $\mathfrak{X}$ there is a positive number ε such that $\rho(a) \geq \varepsilon$, then $a \in \sum \mathfrak{X}^2$.*

The next theorem works only with representations π_ρ of $\mathcal{A}$. It can be considered as a non-commutative version of M. Marshall's extension of the Archimedean Positivstellensatz to noncompact semi-algebraic sets [M2].

Theorem 4. *Assume that (O) and (AB) hold. Let $a \in \mathcal{A}_h$ and $t \in \mathcal{S}$ be such that $y := t^{-1}a(t^*)^{-1} \in \mathfrak{X}$. Then we have:*
(i) If $\pi_\rho(a) \geq 0$ for all irreducible $\mathcal{S}^{-1}$-torsionfree $*$-representations ρ of $\mathfrak{X}$, then for each $\varepsilon > 0$ there exists $s_\varepsilon \in \mathcal{S}_O$ such that $s_\varepsilon^*(a + \varepsilon tt^*)s_\varepsilon \in \sum \mathcal{A}^2$.
(ii) If for any $\varepsilon > 0$ there is an element $s_\varepsilon \in \mathcal{S}$ such that $s_\varepsilon(a + \varepsilon tt^*)s_\varepsilon^* \in \sum \mathcal{A}^2$, then $\pi_\rho(a) \geq 0$ for all $\mathcal{S}^{-1}$-torsionfree $*$-representations ρ of $\mathfrak{X}$.

Proof. (i): Suppose that $\varepsilon > 0$. Since $\pi_\rho(a) \geq 0$, we conclude from equation (5), applied with $b = a$, $r = t$, that $\rho(y) \geq 0$ on $\mathcal{D}(\pi_\rho)$ and by the density of $\mathcal{D}(\pi_\rho)$ on $\mathcal{H}(\rho)$. Thus $y + \varepsilon$ satisfies the assumption of the Positivstellensatz in Remark 4. Therefore, $y + \varepsilon \in \mathfrak{X}^2$, that is, $y + \varepsilon = \sum_i y_i^* y_i$ where $y_i \in \mathfrak{X}$. Proceeding as in the last paragraph of the proof of Theorem 2, there exist elements $s_\varepsilon \in \mathcal{S}_0$ and $a_i \in \mathcal{A}$ such that $y_i t^* = a_i s_\varepsilon^{-1}$ for all i and we get $s_\varepsilon t(y + \varepsilon)(s_\varepsilon t)^* = s_\varepsilon^*(a + \varepsilon tt^*)s_\varepsilon = \sum_i a_i^* a_i \in \sum \mathcal{A}^2$.

(ii): Assume that $s_\varepsilon \in \mathcal{S}$ and $c := s_\varepsilon t(y + \varepsilon)(s_\varepsilon t)^* = s_\varepsilon(a + \varepsilon tt^*)s_\varepsilon^* \in \sum \mathcal{A}^2$. Therefore, since π_ρ is $*$-representation of $\mathcal{A}$, $\pi_\rho(c) \geq 0$. Equation (5), applied with $b = c$, $r = s_\varepsilon t$, $x = y + \varepsilon$, yields that $\rho(y + \varepsilon) \geq 0$ on $\mathcal{D}(\pi_\rho)$ and so on $\mathcal{H}(\rho)$. Since $\varepsilon > 0$ was arbitrary, we have $\rho(y) \geq 0$. Combining the latter with identity (5), now applied with $b = a$, $r = s_\varepsilon$, $x = y$, it follows that $\pi_\rho(a) \geq 0$. □

5. Multi-Graded $*$-Algebras

In this section we assume that the $*$-algebra $\mathcal{A}$ has a multi-degree map $d : \mathcal{A}\setminus\{0\} \to \mathbb{N}_0^k$ satisfying the following conditions for arbitrary non-zero $a, b \in \mathcal{A}$ and $\lambda \in \mathbb{C}$:

$(d1)$ $d(\lambda a) = d(a)$ and $d(a+b) \leq d(a) \vee d(b)$,

$(d2)$ $d(ab) = d(a) + d(b)$,

$(d3)$ $d(a^*) = d(a)$,

where $a + b \neq 0$ in $(d1)$ and we use the following notations for multi-indices $\mathfrak{n}{=}(n_1, \dots, n_k)$, $\mathfrak{m}{=}\{m_1, \dots, m_k) \in \mathbb{Z}^k$: $\mathfrak{n} \vee \mathfrak{m} = (\max(m_1, n_1), \dots, \max(m_k, n_k))$, $\mathfrak{n} \leq \mathfrak{m}$ if $n_1 \leq m_1, \dots, n_k \leq m_k$, $\mathfrak{n} < \mathfrak{m}$ if $n_1 < m_1, \dots, n_k < m_k$.

We extend the map d to a multi-degree map d of $\mathcal{A}\mathcal{S}_O^{-1}$ to $\mathbb{Z}^k$ by putting $d(as^{-1}) = d(a) - d(s)$ for $a \in \mathcal{A}\setminus\{0\}$ and $s \in \mathcal{S}_O$. It is straightforward to check that d is well-defined and that conditions $(d1)$–$(d3)$ hold for the algebra $\mathcal{A}\mathcal{S}_O^{-1}$ as well.

Further, we suppose that the following conditions are valid:

$(A3)$ *$d([a, s]) \leq d(a)$ for all $s \in \mathcal{S}_G$ and $a \in \mathcal{A}_G$.*

$(A4)$ *$as^{-1} \in \mathfrak{X}$ for all $s \in \mathcal{S}_G$ and $a \in \mathcal{A}$ such that $d(a) \leq d(s)$.*

$(A5)$ *For $a \in \mathcal{A}$ and $\mathfrak{n}, \mathfrak{k} \in \mathbb{N}_0^k$ such that $d(a) \leq \mathfrak{n} + \mathfrak{k}$ there exist finitely many elements $b_i, c_i \in \mathcal{A}$ satisfying $d(b_i) \leq \mathfrak{n}$, $d(c_i) \leq \mathfrak{k}$ for all i and $a = \sum_i b_i c_i$.*

Lemma 7. *(i) $d([a, s]) \leq d(a)$ for $s \in \mathcal{S}_G$ and $a \in \mathcal{A}$.*
(ii) $s^{-1}at^{-1} \in \mathfrak{X}$ for $s, t \in \mathcal{S}$ and $a \in \mathcal{A}$ such that $d(a) \leq d(st)$.
(iii) $a(ts)^{-1}b \in \mathfrak{X}$ for $s, t \in \mathcal{S}_G$ and $a, b \in \mathcal{A}$ such that $d(ab) \leq d(ts)$, $d(s){=}d(t)$ and $d(a) \leq d(s)$.
(iv) If $\mathcal{S} = \mathcal{S}_O$, then $as^{-1}b \in \mathfrak{X}$ for $s \in \mathcal{S}$ and $a, b \in \mathcal{A}$ such that $d(ab) \leq d(s)$.

Proof. (i): Let $\mathcal{B}$ denote the set of all $a \in \mathcal{A}$ for which the assertion of (i) is true. By conditions $(d1)$ and $(d3)$, $\mathcal{B}$ is a $*$-invariant linear subspace of $\mathcal{A}$. Suppose that $b_1, b_2 \in \mathcal{B}$ and $s \in \mathcal{S}_G$. Using conditions $(d1)$ and $(d2)$ and the fact that $d([b_l, s]) \leq d(b_l)$, $l = 1, 2$, we obtain

$$d([b_1b_2, s]) = d(b_1[b_2, s]{+}[b_1, s]b_2) \leq (d(b_1){+}d([b_2, s]){\vee}(d([b_1, s]){+}d(b_2)) \leq d(b_1){+}d(b_2) = d(b_1b_2),$$

so $\mathcal{B}$ is a $*$-algebra. Since it contains all generators of $\mathcal{A}$ by assumption $(A3)$, we have $\mathcal{B} = \mathcal{A}$.

(ii): We first treat the case $s = 1$. Suppose that the assertion is valid for some $t \in \mathcal{S}$ and all $a \in \mathcal{A}$. By induction it suffices to show that it holds then for the element ts_j of $\mathcal{S}$, where $s_j \in \mathcal{S}_G$. Let a be an element of $\mathcal{A}$ such that $d(a) \leq d(ts_j)$. By assumption $(A5)$ we can assume without loss of generality that $a = bc$, where $d(b) \leq d(s_j)$ and $d(c) \leq d(t)$. Note that $bs_j^{-1} \in \mathfrak{X}$ by $(A4)$. Since $d(c) \leq d(t)$, we have $d([c, s_j]) \leq d(t) \leq d(ts_j)$ by (i) and hence $[c, s_j](ts_j)^{-1} \in \mathfrak{X}$ and ct^{-1} by the induction hypothesis. Therefore, it follows from the identity

$$bc(ts_j)^{-1} = bs_j^{-1}(ct^{-1} - [c, s_j](ts_j)^{-1})$$

that $bc(ts_j)^{-1} \in \mathfrak{X}$. This completes the proof of (ii) in the case $s = 1$.

Suppose now that $d(a) \leq d(st)$. Again by $(A5)$ we can asumme that $a = bc$, where $d(b) \leq d(s)$ and $d(c) \leq d(t)$. Then $d(b^*) \leq d(s^*)$ by $(d3)$. By the preceding paragraph we have $s^{-1}b = (b^*(s^*)^{-1})^* \in \mathfrak{X}$ and $ct^{-1} \in \mathfrak{X}$, so that $s^{-1}at^{-1} = s^{-1}bct^{-1} \in \mathfrak{X}$.

(iii): It suffices to check that all three summands on the right hand side of the identity

$$a(ts)^{-1}b = s^{-1}abt^{-1} + (s^{-1}a)(t^{-1}[b, t]t^{-1}) + (s^{-1}[a, s])(ts)^{-1}b$$

belong to $\mathfrak{X}$. Indeed, the first one is in $\mathfrak{X}$ by (ii). Since $d([a, s]) \leq d(a)$ by (i) and $d(a) \leq d(s)$ by assumption, the elements $s^{-1}a$ and $s^{-1}[a, s]$ are in $\mathfrak{X}$ by (ii). Since $d[b, t]) \leq d(b) \leq d(t^2) = 2d(t) = d(ts))$ by (i) and by the assumption $d(s) = d(t)$, we have $t^{-1}[b, t]t^{-1} \in \mathfrak{X}$ and $(ts)^{-1}b \in \mathfrak{X}$ once again by (ii). Hence the second and the third summands are also in $\mathfrak{X}$.

(iv): By the assumption $\mathcal{S} = \mathcal{S}_O$, there exist elements $t \in \mathcal{S}$ and $c \in \mathcal{A}$ such that $s^{-1}b = ct^{-1}$. Since then $-d(s){+}d(b) = d(c){-}d(t)$, we have $d(ac) = d(a){+}d(c) \leq d(t)$ and hence $as^{-1}b = act^{-1} \in \mathfrak{X}$ by (ii). ☐

Lemma 8. *Let ρ be a $*$-representation of a $*$-algebra $\mathcal{B}$ and $b \in \mathcal{B}$. If $\rho((b^*b)^m) = 0$ for some $m \in \mathbb{N}$, then $\rho(b) = 0$.*

Proof. Upon multiplying by some appropriate power $(b^*b)^k$ we can assume that $m = 2^n$ for some $n \in \mathbb{N}_0$. If $m = 1$, then $||\rho(b)\varphi||^2 = \langle \rho(b^*b)\varphi, \varphi\rangle = 0$ for $\varphi \in \mathcal{D}(\rho)$ and hence $\rho(b) = 0$. By induction the same reasoning shows that the assertion holds for all numbers m of the form $m = 2^n$, where $n \in \mathbb{N}_0$. □

Lemma 9. *Let $c \in \mathcal{A}$, $s \in \mathcal{S}_G$ and $m \in \mathbb{N}$. Suppose that $d(c) \leq (2m-1)(d(s)-d(c))$. Then we have $((cs^{-1})^*(cs^{-1}))^m \in \mathfrak{X}s^{-1}$ and $\rho_s(cs^{-1}) = 0$ for each $*$-representation ρ_s of the quotient $*$-algebra $\mathfrak{X}_s = \mathfrak{X}/\mathcal{J}_s$.*

Proof. First note that $cs^{-1} \in \mathfrak{X}$ by Lemma 7(ii), since $d(c) \leq d(s)$ by assumption.

We define a sequence of multi-indices $\mathfrak{n}_j{=}(n_{j1}, \dots, , n_{jk})$, $j{=}1, \dots, m$. If $m{=}1$, put $\mathfrak{n} = 2d(c)$. Now let $m \geq 2$. Fix $l \in \{1, \dots, k\}$. If $d(s)_l \geq 2d(c)_l$, we set $n_{jl} = 2d(c)_l$ for $j{=}1, \dots, m$. Suppose that $d(s)_l \leq 2s(c)_l$. Then there exists a number $m_l \in \{2, \dots, m\}$ such that

$$(2m_l - 3)(d(s)_l - d(c)_l) \leq d(c)_l \leq (2m_l - 1)(d(s)_l - d(c)_l) \tag{7}$$

Define $n_{1l} = d(s)_l$, $n_{jl} = 2d(c)_l$ if $m_l \leq j \leq m$ and

$$n_{jl} = 2(j-1)(d(s)_l{-}d(c)_l) + d(s)_l \quad \text{if} \quad 2 \leq j \leq m_l{-}1.$$

Using the preceding definitions we verify that

$$d(c) \leq \mathfrak{n}_j \leq 2d(c) \text{ for } j = 1, \dots, m, \tag{8}$$

$$2d(c){-}\mathfrak{n}_{j-1} + \mathfrak{n}_j \leq 2d(s) \text{ for } j = 2, \dots, m. \tag{9}$$

Indeed, for $j{=}1$, we have $d(c)_l \leq n_{1l} = d(s)_l \leq 2d(c)_l$. If $2 \leq j \leq m_l{-}1$, using the first inequality of (7) we derive

$$n_{jl} \leq 2(m_l{-}2)(d(s)_l - d(c)_l) + d(s)_l = (2m_l{-}3)(d(s)_l - d(c)_l) + d(c)_l \leq 2d(c)_l$$

and from the definition of n_{jl} we obtain

$$n_{jl} \geq n_{2l} = 2(d(s)_l - d(c)_l) + d(s)_l \geq d(c)_l.$$

This proves (8). If $2 \leq j \leq m_l{-}1$, we have $2d(c)_l{-}n_{j-1,l}{+}n_{jl} = 2d(s)_l$ by the above definitions. If $j = m_l$, then the corresponding definitions and the second inequality of (7) yield

$$\begin{aligned} 2d(c)_l - n_{j-1,l} + n_{jl} &= 2d(c)_l - 2(m_l{-}2)(d(s)_l - d(c)_l) - d(s)_l + 2d(c)_l \\ &= d(c)_l - (2m_l{-}1)(d(s)_l - d(c)_l) + 2d(s)_l \leq 2d(s)_l \end{aligned}$$

which proves (9).

Now we write the element $((cs^{-1})^*(cs^{-1}))^m$ of $\mathfrak{X}$ in the form

$$((cs^{-1})^*(cs^{-1}))^m = t^{-1}A_1(ts)^{-1}A_2(ts)^{-1} \cdots A_m s^{-1}, \tag{10}$$

where $t := s^*$ and $A_1 = \cdots = A_m := c^*c$. Note that $d(s) = d(t)$ and $d(A_j) = 2d(c) \leq d(ts)$.

If $m{=}1$, then $d(A_1) = 2d(c) \leq d(s){=}d(t)$ and hence $t^{-1}A_1 \in \mathfrak{X}$ by Lemma 7(ii).

Now suppose that $m \geq 2$. For $j{=}1, \dots, m{-}1$, set $\mathfrak{k}_j := 2d(c){-}\mathfrak{n}_j$. By the second inequality of (8), we have $\mathfrak{k}_j \in \mathbb{N}_0^k$. By definition, $\mathfrak{n}_j + \mathfrak{k}_j = 2d(c) = d(A_j)$. Therefore, by condition (A5) we can write the element A_j of $\mathcal{A}$ as a finite sum $A_j = \sum_i b_{ji}c_{ji}$ of elements $b_{ji}, c_{ji} \in \mathcal{A}$ such that $d(b_{ji}) \leq \mathfrak{n}_j$ and $d(c_{ji}) \leq \mathfrak{k}_j$. Since $\mathfrak{n}_1{=}d(t)$ by definition, $t^{-1}b_{1i} \in \mathfrak{X}$ by Lemma 7(ii). If $j{=}2, \dots, m{-}1$, then we have $\mathfrak{k}_{j-1} + \mathfrak{n}_j = 2d(c) - \mathfrak{n}_{j-1} + \mathfrak{n}_j \leq 2d(s) = d(ts)$ by (9) and $\mathfrak{k}_j = 2d(c) - \mathfrak{n}_j \leq d(c) \leq d(s)$ by the first inequality of (8). Therefore, Lemma 7(iii) applies and yields that $c_{j-1,i}(ts)^{-1}b_{j,i'} \in \mathfrak{X}$. Finally, we have $(ts)^{-1}A_m \in \mathfrak{X}$, since $\mathfrak{n}_m = 2d(c){=}d(A_m)$ by construction.

In the preceding two paragraphs we have shown that $t^{-1}A_1(ts)^{-1}A_2(ts)^{-1} \cdots A_m \in \mathfrak{X}$. Therefore, by (10) the element $((cs^{-1})^*(cs^{-1}))^m$ belongs to $\mathfrak{X}s^{-1} \subseteq \mathcal{J}_s$, so that $\rho_s(((cs^{-1})^*(cs^{-1}))^m) = 0$. The second assertion follows from Lemma 8 applied to $b = cs^{-1}$. □

Remark 5. The preceding proof shows that the assertion of Lemma 9 is valid for $s \in \mathcal{S}$ (rather than $s \in \mathcal{S}_G$) provided that $a(s^*s)^{-1}b \in \mathfrak{X}$ for all $a, c \in \mathcal{A}$ satisfying $d(a) \leq d(s)$ and $d(ab) \leq 2d(s)$.

The next three propositions contains results about elements which are annihilated by the representations ρ_s of the quotient $*$-algebras $\mathfrak{X}_s$.

Proposition 1. *Let $s, t \in \mathcal{S}_G$ and $a \in \mathcal{A}$ be such that $d(a) < d(st)$. Then $\rho_s(s^{-1}at^{-1}) = \rho_s(t^{-1}as^{-1}) = 0$ for any $*$-representation ρ_s of the $*$-algebra $\mathfrak{X}_s = \mathfrak{X}/\mathcal{J}_s$.*

Proof. The assumption $d(a) < d(st)$ implies that $d(a)_l < d(s)_l + d(t)_l$ for $l = 1, \dots, k$. We choose $\mathfrak{n}, \mathfrak{k} \in \mathbb{N}_0^k$ such that $n_l + k_l = a_l$, $n_l \leq d(t)_l$ and $k_l < d(s)_l$ for $l{=}1, \dots, k$. Since $d(a) = \mathfrak{n} + \mathfrak{k}$, by condition $(A5)$ we can write $a = \sum_i b_i c_i$, where $b_i, c_i \in \mathcal{A}$, $d(b_i) \leq \mathfrak{n}$ and $d(c_i) \leq \mathfrak{k}$. Since $d(c_i)_l \leq k_l < d(s)_l$, there is a number $m \in \mathbb{N}$ such $m(d(s){-}d(c_i)) \geq d(c_i)$ for all i. Then we have $t^{-1}b_i, c_i s^{-1} \in \mathfrak{X}$ by Lemma 7(ii) and $\rho_s(c_i s^{-1}) = 0$ by Lemma 9, so that $\rho_s(t^{-1}as^{-1}) = \sum_i \rho_s(t^{-1}b_i)\rho_s(c_i s^{-1}) = 0$. □

Proposition 2. *Suppose that $\mathcal{S} = \mathcal{S}_O$. Let $s{=}s_1 \dots s_p \in \mathcal{S}$ and $t{=}s_{p+1} \dots s_{p+q} \in \mathcal{S}$, where $s_l \in \mathcal{S}_G$ for $l{=}1, \dots, p{+}q$. If $a \in \mathcal{A}$ and $d(a) < d(st)$, then we have $\rho_{s_l}(s^{-1}at^{-1}) = \rho_{s_l}(t^{-1}as^{-1}) = 0$ for each $*$-representation ρ_{s_l} of the $*$-algebra $\mathfrak{X}_{s_l} = \mathfrak{X}/\mathcal{J}_{s_l}$, $l{=}1, \dots, p{+}q$.*

Proof. Let us carry out the proof of $\rho_{s_l}(t^{-1}as^{-1}) = 0$ for $l{=}1, \dots, p$. The other assertions are derived in a similar manner. We argue as in the preceding proof of Proposition 1 and retain the notation used therein. Since $\mathcal{S} = \mathcal{S}_O$, it follows from Lemma 7(iv) and Remark 5 that the assertion of Lemma 9 is valid for s and c_i, that is, we have $((c_i s^{-1})^*(c_i s^{-1}))^m \in \mathfrak{X}s^{-1} \subseteq \mathcal{J}_{s_l}$. Hence $\rho_{s_l}(c_i s^{-1}) = 0$ by Lemma 8 which in turn implies that $\rho_{s_l}(t^{-1}as^{-1}) = \sum_i \rho_{s_l}(t^{-1}b_i)\rho_{s_l}(c_i s^{-1}) = 0$. □

For the next proposition we need one more notation. Let $s \in \mathcal{S}$, $r \in \mathcal{S}_G$ and $a \in \mathcal{A}$. We say that r is a factor of s if there are elements $s_1, \dots, s_p \in \mathcal{S}_G$ and $i \in \{1, \dots, p\}$ such that $s = s_1 \dots s_p$ and $r = s_i$. We shall write $a <_r s$ if r is a factor of s and there are multi-indices $\mathfrak{r}, \mathfrak{n} \in \mathbb{N}_0^k$ such that $d(a) = \mathfrak{r} + \mathfrak{n}$, $\mathfrak{r} < d(r)$ and $\mathfrak{n} \leq d(s){-}d(r)$.

Proposition 3. *Suppose that $\mathcal{S} = \mathcal{S}_O$. Let $s, t \in \mathcal{S}$, $r \in \mathcal{S}_G$ and $a \in \mathcal{A}$. Assume that r is a factor of s or a factor of t. If $a <_r st$, then $\rho_r(s^{-1}at^{-1}) = \rho_r(t^{-1}as^{-1}) = 0$ for each $*$-representation ρ_r of $\mathfrak{X}_r = \mathfrak{X}/\mathcal{J}_r$.*

Proof. The proof follows by some modifications in the proofs of Lemma 9 and Proposition 1. We explain this for the proof of $\rho_r(t^{-1}as^{-1}) = 0$ and in the case where r is a factor of s, say $s = s_1 \dots s_p$ and $r = s_i$.

First we modify the proof of Lemma 9. Let c be an element of $\mathcal{A}$ such that $c <_r s$. We write $d(c) = \mathfrak{r} + \mathfrak{n}$ with $\mathfrak{r} < d(r)$ and $\mathfrak{n} \leq d(s){-}d(r)$. Since $\mathfrak{r} < d(r)$, there exists an $m \in \mathbb{N}$ such that $\mathfrak{r} \leq (2m-1)(d(r){-}\mathfrak{r})$. We construct a sequence of multi-indices $\mathfrak{n}_j$ as in the proof of Lemma 9 with $d(c)$ replaced by $\mathfrak{r}$ and $d(s)$ replaced by $d(r)$ therein. Then equations (8) and (9) yield $\mathfrak{n}_j \leq 2\mathfrak{r}$ and $2\mathfrak{r} - \mathfrak{n}_{j-1} + \mathfrak{n}_j \leq 2d(r)$. Put $\mathfrak{k}_j := 2\mathfrak{r} - \mathfrak{n}_j$. We now decompose $A_j = c^*c$ as a finite sum $A_j = \sum_i b_{ji}c_{ji}$ with $d(b_{ji}) \leq \mathfrak{n}_j + d(c){-}d(r)$ and $d(c_{ji}) \leq \mathfrak{k}_j + d(c){-}d(r)$. Then we obtain

$$d(c_{j-1,i}b_{ji'}) \leq \mathfrak{k}_{j-1} + \mathfrak{n}_j + 2d(c) - 2d(r) = 2\mathfrak{r} - \mathfrak{n}_{j-1} + \mathfrak{n}_j + 2d(c) - 2d(r) \leq 2d(c) \leq d(st).$$

Since we assumed that $\mathcal{S} = \mathcal{S}_O$, Lemma 7(iv) applies and yields that $c_{j-1,i}(ts)^{-1}b_{ji'} \in \mathfrak{X}$. In a similar manner we obtain that $t^{-1}b_{1i} \in \mathfrak{X}$. Recall that $\mathfrak{n}_m = 2\mathfrak{r}$ and $\mathfrak{k}_m = 0$ by construction. Therefore we have $d(c_{mi}) \leq d(c){-}d(r)$ and so $d(rc_{mi}) \leq d(c) \leq d(s)$. Employing again Lemma 7(iv) we get $rc_{mi}s^{-1} \in \mathfrak{X}$ and so $c_{mi}s^{-1} = r^{-1}(rc_{mi}s^{-1}) \in \mathcal{J}_r$. Combining the latter with (10) it follows that $((cs^{-1})^*(cs^{-1}))^m \in \mathcal{J}_r$. Hence we obtain $\rho_r(cs^{-1}) = 0$ by Lemma 8.

Since $a <_r d(st)$, as in the proof of Proposition 1 we decompose $d(a) = \mathfrak{n} + \mathfrak{k}$, where $\mathfrak{n} \leq d(t)$, $\mathfrak{r} < \mathfrak{k} \leq d(s)$, and $\mathfrak{r} < d(r)$. By (A6) we can write and $a = \sum b_l c_l$ with $d(b_l) \leq \mathfrak{n}$ and $d(c_l) \leq \mathfrak{k}$. Since r is a factor of s, we have $c_i <_r s$ and hence $\rho_r(c_l s^{-1}) = 0$ as shown in the preceding paragraph. Because of $t^{-1}b_l \in \mathfrak{X}$ by Lemma 7(ii), we conclude that $\rho_r(t^{-1}as^{-1}) = 0$. □

6. Application: A Strict Positivstellensatz for the Weyl Algebra

Throughout this section $\mathcal{A}$ denotes the Weyl algebra $\mathcal{W}(1)$, that is, $\mathcal{A}$ is the unital $*$-algebra with hermitian generators p and q and defining relation

$$pq - qp = -i1. \tag{11}$$

It is well-known that this commutation relation is satisfied by the self-adjoint operators

$$(P_0\varphi)(t) = -i\varphi'(t) \;\text{ and }\; (Q_0\psi)(t) = t\psi(t),\; t \in \mathbb{R},$$

on the Hilbert space $L^2(\mathbb{R})$. The pair (P_0, Q_0) is called *Schrödinger pair* and the corresponding $*$-representation π_0 of the $*$-algebra $\mathcal{A}$ is the *Schrödinger representation.* That is,

$$(\pi_0(p)\varphi)(t) = -i\varphi'(t) \text{ and } (\pi_0(q)\varphi)(t) = t\varphi(t) \;\text{ for }\; \varphi \in \mathcal{D}(\pi_0){=}\mathcal{S}(\mathbb{R}) \subseteq \mathcal{H}(\pi_0){=}L^2(\mathbb{R}).$$

We fix two non-zero reals α and β and put

$$\mathcal{S}_g = \{s_1 = p - \alpha i,\; s_2 = q - \beta i\},\; \mathcal{S}_G = \mathcal{S}_g \cup \mathcal{S}_g^*,\; \mathfrak{X}_G = \mathcal{S}_G^{-1},\; \mathcal{A}_G = \{p,\; q\}.$$

From the relation (11) it follows immediately that the $*$-monoid $\mathcal{S}$ generated by $\mathcal{S}_G$ is an Ore set, that is, we can assume that $\mathcal{S}{=}\mathcal{S}_O$. The unital $*$-subalgebra $\mathfrak{X}$ of $\mathcal{A}\mathcal{S}_O^{-1}$ is generated by $x := s_1^{-1}$ and $y := s_2^{-1}$. From (11) we easily derive the following relations in the $*$-algebra $\mathfrak{X}$:

$$x - x^* = 2i\alpha\; x^*x,\quad y - y^* = 2i\beta\; y^*y, \tag{12}$$

$$xx^* = x^*x,\; yy^* = y^*y, \tag{13}$$

$$xy - yx = -ixy^2x = -iyx^2y,\; xy^* - y^*x = -ix(y^*)^2x = -iy^*x^2y^*. \tag{14}$$

Lemma 10. *With the preceding definitions, conditions* (O), (IA) *and* (AB) *are fulfilled.*

Proof. Let us prove (AB). Using relations (12) it follows that

$$1 - \alpha^2x^*x = (1 + i\alpha x)^*(1 + i\alpha x) \text{ and } 1 - \beta^2y^*y = (1 + \beta iy)^*(1 + \beta iy) \tag{15}$$

are in $\sum \mathfrak{X}^2$, so conclude that $\mathfrak{X} = \mathfrak{X}_b$. This means that $\mathfrak{X}$ is algebraically bounded, so (AB) is satisfied.

Condition (IA) is easily derived from relations (12)–(14) and condition (O) follows from (IA) according to Lemma 3. $\square$

Lemma 11. *Let* $\gamma \in \mathbb{R}$ *and let* z *be a bounded normal operator on a Hilbert* $\mathcal{H}$ *such that* $z - z^* = 2\gamma iz^*z$ *and* $\ker z = \{0\}$. *Then* $A := z^{-1} + i\gamma I$ *is a self-adjoint operator on* $\mathcal{H}$.

Proof. First we note that $\ker z^* = \{0\}$, because z is normal. Since $z^* = z(I - 2\gamma iz^*)$ and $z = z^*(I{+}2\gamma iz)$, we have $\mathcal{D}((z^*)^{-1}){=}z^*\mathcal{H}{=}z\mathcal{H}{=}\mathcal{D}(z^{-1})$. Further, from the identity $z^* = z(I{-}2\gamma iz^*)$ we get $z^{-1}z^* = I - 2i\gamma z^*$ on $\mathcal{H}$. For $\varphi = z^*\psi \in \mathcal{D}((z^*)^{-1})$ we obtain $z^{-1}\varphi = z^{-1}z^*\psi = \psi - 2i\gamma z^*\psi = (z^*)^{-1}\varphi - 2i\gamma\varphi$, that is, $z^{-1} \supseteq (z^*)^{-1} - 2\gamma iI$. Because $\mathcal{D}((z^*)^{-1}) = \mathcal{D}(z^{-1})$ as noticed above, it follows that $z^{-1} = (z^*)^{-1} - 2i\gamma I$. Using the latter identity we derive

$$A = z^{-1} + i\gamma I = (z^*)^{-1} - i\gamma I = (z^{-1})^* - i\gamma I = (z^{-1} + i\alpha I)^* = A^*. \qquad \square$$

The assertion of the next proposition describes Schrödinger pairs in terms of resolvents. A slightly different characterization of this kind has been first obtained in [B].

Proposition 4. *Suppose that* x *and* y *are closed linear operators on a Hilbert space* $\mathcal{H}$ *with trivial kernels (that is,* $\ker x = \ker y = \{0\}$*) satisfying equations (12)–(14). Then*

$$P = x^{-1} + i\alpha I \text{ and } Q = y^{-1} + \beta iI \tag{16}$$

are self-adjoint operators on $\mathcal{H}$ *and the pair* (P, Q) *is unitarily equivalent to a direct sum of Schrödinger pairs* (P_0, Q_0) *on* $L^2(\mathbb{R})$.

Proof. The self-adjointness of operators P and Q follows from Lemma 11.

From the first equations of (14) we conclude that $xy\mathcal{H} = yx\mathcal{H}$. Let us denote this vector space by $\mathcal{D}$. Since $\mathcal{D}(P){=}x\mathcal{H}$ and $\mathcal{D}(Q){=}y\mathcal{H}$ by (16), we have $\mathcal{D} \subseteq \mathcal{D}(PQ) \cap \mathcal{D}(QP)$.

We show that $PQ\varphi - QP\varphi = -i\varphi$ for $\varphi \in \mathcal{D}$. Indeed, if $\varphi = yx\psi$, then by the first equations of (14) we derive

$$\begin{aligned} PQ\varphi - QP\varphi &= (P - i\alpha)(Q - i\beta)\varphi - (Q - i\beta)(P - i\alpha)\varphi \\ &= (P - i\alpha)(Q - i\beta)yx\psi - (Q - i\beta)(P - i\alpha)xy(I + iyx)\psi \\ &= \psi - (I + iyx)\psi = i\varphi. \end{aligned}$$

Moreover, from the definitions (16) we obtain $(P-i\alpha)(Q-i\beta)\mathcal{D}=(P-i\alpha)(Q-i\beta)yx\mathcal{H}=\mathcal{H}$ and $(Q-i\beta)(P-i\alpha)\mathcal{D}=(Q-i\beta)(P-i\alpha)xy\mathcal{H}=\mathcal{H}$.

By the preceding we have shown that P and Q satisfy the assumptions of a theorem by T. Kato [K2]. The assertion of this theorem states that

$$e^{i\lambda P}e^{i\mu Q}=e^{i\lambda\mu}e^{i\mu Q}e^{i\lambda P} \tag{17}$$

for nonnegative reals λ and μ. That (17) holds for nonnegative reals obviously implies that (17) is fulfilled for arbitrary reals λ and μ. Thus, P and Q are self-adjoint operators satisfying the Weyl relation. By the Stone–von Neumann uniqueness theorem (see e.g. [Pu], Theorem 4.3.1), the pair (P,Q) is is unitarily equivalent to a direct sum of Schrödinger pairs (P_0,Q_0). $\square$

Proposition 5. *Suppose ρ is an $\mathcal{S}^{-1}$-torsionfree $*$-representation of the $*$-algebra $\mathfrak{X}$. Then the $*$-representation π_ρ of $\mathcal{A}$ is unitarily equivalent to a direct sum of Schrödinger representations.*

Proof. Since the $*$-algebra $\mathfrak{X}$ is algebraically bounded by Lemma 10, all operators of $\rho(\mathfrak{X})$ are bounded. The operators $\rho(x)$ and $\rho(y)$ satisfy the relations (12)–(14) and have trivial kernels because ρ is torsionfree. Therefore, by Proposition 4 the pair (P,Q) defined by (16) (with x and y replaced by $\rho(x)$ and $\rho(y)$, respectively) is unitarily equivalent to a direct sum of Schrödinger pairs. The map $\rho\to\pi_\rho$ according to Theorem 1 respects unitary equivalences and direct sums, so it suffices to prove the assertion in the case when $P=P_0$ and $Q=Q_0$ on the Hilbert space $L^2(\mathbb{R})$. By (2) the domain $\mathcal{D}_\rho=\mathcal{D}(\pi_\rho)$ is the intersection of ranges of all finite products of operators $(P-\alpha i)^{-1}=\rho(x)=\rho(s_1^{-1})$, $(Q-\beta i)^{-1}=\rho(y)=\rho(s_2^{-1})$ and their adjoints. Hence $\mathcal{D}_\rho=\mathcal{D}(\pi_\rho)$ is the Schwartz space $\mathcal{S}(\mathbb{R})$ and for $\varphi\in\mathcal{D}(\pi_\rho)$ we have

$$\pi_\rho(p-\alpha i)\varphi=\pi_\rho(s_1)\varphi=\rho(s_1^{-1})^{-1}\varphi=\rho(x)^{-1}\varphi=(P-\alpha i)\varphi=-i\varphi'-\alpha i\varphi,$$

so $\pi_\rho(p)\varphi=-i\varphi'$. Similarly, $\pi_\rho(q)\varphi=t\varphi$. That is, π_ρ is the Schrödinger representation. $\square$

Now let c be an arbitrary nonzero element of the Weyl algebra $\mathcal{A}$. Because $\{p^nq^k;k,n\in\mathbb{N}_0\}$ and $\{q^np^k;k,n\in\mathbb{N}_0\}$ are vector space bases of $\mathcal{A}$, we can write c as

$$c=\sum_{j=0}^{d_1}\sum_{l=0}^{d_2}\gamma_{jl}p^jq^l=\sum_{n=0}^{d_2}f_n(p)q^n=\sum_{k=0}^{d_1}g_k(q)p^k, \tag{18}$$

where γ_{jl} are complex numbers and $f_n(p)\in\mathbb{C}[p]$, $g_k(q)\in\mathbb{C}[q]$ are polynomials all of them uniquely determined by c. We choose d_1 and d_2 such that there are numbers $j_0,l_0\in\mathbb{N}_0$ for which $\gamma_{d_1,l_0}\neq 0$ and $\gamma_{j_0,d_2}\neq 0$. Set $d(c)=(d_1,d_2)$. It is easily checked that d defines a multi-degree on $\mathcal{A}$ satifying conditions $(d1)$–$(d3)$ and $(A3)$–$(A5)$. Note that $f_{d_2}\neq 0$ and $g_{d_1}\neq 0$.

Theorem 5. *Let $c=c^*$ be a nonzero element of the Weyl algebra $\mathcal{A}$ with multi-degree $d(c)=(2m_1,2m_2)$, where $m_1,m_2\in\mathbb{N}_0$. Suppose that:*
(I) There exists a bounded self-adjoint operator $T>0$ on $L^2(\mathbb{R})$ such that $\pi_0(c)\geq T$.
(II) $\gamma_{2m_1,2m_2}\neq 0$ and both polynomials f_{2m_2} and g_{2m_1} are positive on the real line.
Then there exists an element $s\in\mathcal{S}$ such that $s^*cs\in\sum\mathcal{A}^2$.

Proof. Recall that $\mathcal{S}=\mathcal{S}_O$ and all results from Sections 4 and 5 apply, because the corresponding assumptions are fulfilled. Set $t:=s_2^{m_2}s_1^{m_1}$. Since $d(c)=(2m_1,2m_2)=d(t^2)$, it follows from Lemma 7(ii) that $z:=t^{-1}c(t^*)^{-1}=x^{m_1}y^{m_2}c(y^*)^{m_2}(x^*)^{m_1}$ is in $\mathfrak{X}$.

The assertion will follow from Theorem 3 once assumptions (i) and (ii) therein are established. Assumption (i) is a consequence of assumption (I), since the only irreducible $*$-representations π_ρ is the Schrödinger representation π_0 by Proposition 5.

We prove that $\rho_{s_1}(z)>0$ for each $*$-representation ρ_{s_1} of $\mathfrak{X}_{s_1}$. (By Theorem 3 we could assume that ρ_s is irreducible, but this does not simplify our reasoning.) If $k<2m_1$, then $d(g_k(q)p^k)_1<2m_1$, so that $g_k(q)p^k<_{s_1}t^*t$ and hence $\rho_{s_1}(t^{-1}g_k(q)p^k(t^*)^{-1})=0$ by Proposition 3. Likewise, $d(p^{m_1}g_{2m_1}(q)p^{m_1}-g_{2m_1}(q)p^{2m_1})_1<2m_1$ and so $\rho_{s_1}(t^{-1}(p^{m_1}g_{2m_1}(q)p^{m_1}-g_{2m_1}(q)p^{2m_1})(t^*)^{-1})=0$ again by Proposition 3. Therefore, by (18) we have

$$\rho_{s_1}(z)=\rho_{s_1}(x^{m_1}y^{m_2}p^{m_1}g_{2m_1}(q)p^{m_1}(y^*)^{m_2}(x^*)^{m_1}). \tag{19}$$

Since $xy = yx(1-ixy)$ by (14), $x^{m_1}y^{m_2} - y^{m_2}x^{m_1}$ and $(y^*)^{m_2}(x^*)^{m_1}) - (x^*)^{m_1}(y^*)^{m_2}$ are linear combinations of terms r^{-1}, where $r \in \mathcal{S}$ and $d(r) > (m_1, m_2)$. Hence from Proposition 3 we get

$$\rho_{s_1}(x^{m_1}y^{m_2}p^{m_1}g_{2m_1}(q)p^{m_1}(y^*)^{m_2}(x^*)^{m_1} - y^{m_2}x^{m_1}p^{m_1}g_{2m_1}(q)p^{m_1}(x^*)^{m_1}(y^*)^{m_2}) = 0. \tag{20}$$

From the relation $xp = 1+\alpha ix$ it follows that $x^{m_1}p^{m_1} - 1$ and $p^{m_1}(x^*)^{m_1} - 1$ are linear combinations of $x^j = s_1^{-j}$, where $1 \leq j \leq m_1$. Therefore, we have

$$\rho_{s_1}(z) = \rho_{s_1}(y^{m_2}x^{m_1}p^{m_1}g_{2m_1}(q)p^{m_1}(x^*)^{m_1}(y^*)^{m_2} - y^{m_2}g_{2m_1}(q)(y^*)^{m_2}) = 0 \tag{21}$$

by Proposition 3. (All facts derived above from Proposition 3 can be also verified directly by using the commutation rules between $p, q, x,$ and y.) Combining equations (19)–(21) we obtain $\rho_{s_1}(z) = \rho_{s_1}((yy^*)^{m_2}g_{2m_1}(q))$. Let us write the polynomial g_{2m_2} as $g_{2m_1}(q) = \sum_{l=0}^{2m_2}\gamma_l q^l$. Clearly, $\gamma_{2m_2} = \gamma_{2m_1,2m_2}$. Since $q = y^{-1} + \beta i$ by the definition of $y = s_2^{-1}$ and $y^* = y(1-2\beta iy^*)$ by (12),

$$\rho_{s_1}(z) = \rho_{s_1}((yy^*)^{m_2}g_{2m_1}(q)) = (I - 2\beta i\rho_{s_1}(y)^*)^{m_2}\sum_{l=0}^{2m_2}\gamma_l(I+\beta i\rho_{s_1}(y))^l\rho_{s_1}(y)^{2m_2-l} \tag{22}$$

is a polynomial, say $h(\rho_{s_1}(y))$, of the normal operator $\rho_{s_1}(y)$ and its adjoint. Hence the spectrum of $\rho_{s_1}(z)$ is the set of numbers $h(\mathsf{y})$, where y is in the spectrum of $\rho_{s_1}(y)$. Since $y - y^* = 2\beta iy^*y$ by (12), y belongs to the circle $\mathsf{y} - \bar{\mathsf{y}} = \beta i\bar{\mathsf{y}}\mathsf{y}$ of the complex plane. If $\mathsf{y} = 0$, then $h(0) = \gamma_{2m_2} = \gamma_{2m_1,2m_2} > 0$ by assumption (II). If y is a nonzero number of this circle, then y is of the form $(\mathsf{q} - \beta i)^{-1}$ with $\mathsf{q} \in \mathbb{R}$. Inserting this into (22), we compute $h(\mathsf{y}) = (\mathsf{y}\bar{\mathsf{y}})^{\mathsf{m}_2}\mathsf{g}_{2\mathsf{m}_1}(\mathsf{q})$. Since $g_{2m_1}(\mathsf{q}) > 0$ by assumption (II), we get $h(\mathsf{y}) > 0$. Thus we have shown that the spectrum of the normal operator $\rho_{s_1}(z)$ is contained in $(0, +\infty)$, so that $\rho_{s_1}(z) > 0$.

A similar reasoning using the positivity of f_{2m_2} instead of that of g_{2m_1} yields $\rho_{s_2}(z) > 0$. Hence assumption (ii) of Theorem 3 is satisfied. □

7. A Resolvent Approach to Integrable Representations of the Enveloping Algebra of the $ax + b$-Group

Throughout this and the next section we denote by G the affine group of the line, that is, $G = \{(e^\gamma, \delta); \gamma, \delta \in \mathbb{R}\}$ with multiplication rule $(e^{\gamma_1}, \delta_1)(e_2^\gamma, \delta_2) = (e^{\gamma_1+\gamma_2}, e^{\gamma_1}\delta_2 + \delta_1)$ and by $\mathfrak{g}$ the Lie algebra of the Lie group G. Recall that $\mathfrak{g}$ has a vector space basis $\{\mathsf{a}, \mathsf{b}\}$ satisfying the commutation relation $[\mathsf{a}, \mathsf{b}] = \mathsf{b}$. The exponential map exp of $\mathfrak{g}$ into G is given by $\exp \gamma\mathsf{a} = (e^\gamma, 0)$ and $\exp \gamma\mathsf{b} = (1, \gamma)$, where $\gamma \in \mathbb{R}$.

We need a few notions on Lie group representations (see e.g. [S1], Chapter 10, or [W], Chapter 4, for more details). By a unitary representation of G we mean a strongly continuous homomorphism U of G into the unitary group of a Hilbert space $\mathcal{H}(U)$ and by dU we denote the associated $*$-representation of the enveloping algebra $\mathcal{E}(\mathfrak{g})$ of the Lie algebra $\mathfrak{g}$ on the dense vector space $\mathcal{D}^\infty(U)$ of C^∞-vectors of U. If $\mathsf{c} \in \mathfrak{g}$, then $\partial U(\mathsf{c})$ denotes the infinitesimal generator of the unitary group $U(e^{\gamma\mathsf{c}})$, that is, $e^{\gamma\partial U(\mathsf{c})} = U(e^{\gamma\mathsf{c}})$, $\gamma \in \mathbb{R}$, and we have $dU(\mathsf{c})\varphi = \partial U(\mathsf{c})\varphi$ for $\varphi \in \mathcal{D}^\infty(U)$. Note that the operator $i\partial U(\mathsf{c})$ is self-adjoint.

The next proposition and its subsequent theorem characterize integrable representations of the Lie algebra $\mathfrak{g}$ in terms of resolvents of the two generators.

Proposition 6. *Suppose that U is a unitary reresentation of G. Let α and β be real numbers such that $|\alpha| > 1$, $\beta \neq 0$, and set $x_0 = (A - \alpha i)^{-1}$, $x_1 = (A - (\alpha+1)i)^{-1}$ and $y = (B - \beta i)^{-1}$, where $A := i\partial U(\mathsf{a})$ and $B := i\partial U(\mathsf{b})$. Then we have the relations*

$$x_0 - x_0^* = 2\alpha i\ x_0^*x_0 = 2\alpha i\ x_0x_0^*,\ \ y - y^* = 2\beta i\ y^*y = 2\beta i\ yy^*, \tag{23}$$

$$x_0 - x_1 = -ix_1x_0 = -ix_0x_1, \tag{24}$$

$$x_0y - yx_1 = -\beta yx_1x_0y. \tag{25}$$

Proof. Equations (23) and (24) follow easily from the definitions of x_0, x_1 and y.

We prove the commutation relation (25). From the relation $e^{-\gamma\mathsf{a}}e^{-\delta e^\gamma\mathsf{b}} = e^{-\delta\mathsf{b}}e^{-\gamma\mathsf{a}}$ in the group G it follows that

$$e^{i\gamma A}e^{i\delta e^\gamma B} = e^{i\delta B}e^{i\gamma A}\quad \text{for}\quad \gamma, \delta \in \mathbb{R}. \tag{26}$$

First assume that $\beta < 0$. Then, if C is a self-adjoint operator, we have (see e.g. [K1], p. 482)

$$(C-\beta i)^{-1} = -i\int_0^\infty e^{\beta\lambda} e^{i\lambda C} d\lambda.$$

Multiplying (26) by $e^{\beta\lambda}$ and integrating on $[0,+\infty)$ by using the preceding formula we get

$$e^{i\gamma A}(e^\gamma B-\beta i)^{-1} = (B-\beta i)^{-1} e^{i\gamma A} \text{ for } \gamma\in\mathbb{R}. \tag{27}$$

Applying the involution to (27) and multiplying then by $e^{-\gamma}$ it follows that formula (27) holds in the case $\beta>0$ as well. We now apply both sides of (27) to a vector $\varphi\in\mathcal{D}(A)$ and differentiate at $\gamma=0$. Then we obtain

$$iA(B-\beta i)^{-1}\varphi - B(B-\beta i)^{-2}\varphi = (B-\beta i)^{-1} iA\varphi. \tag{28}$$

Since $y=(B-\beta i)^{-1}$, the latter yields $(A-\alpha i)y\varphi - \beta y^2\varphi = y(A-(\alpha{+}1)i)\varphi$. If $\psi\in\mathcal{H}$, then $\varphi := x_1\psi\in\mathcal{D}(A)$ and so $(A-\alpha i)yx_1\psi - \beta y^2 x_1\psi = y\psi$. Multiplying by x_0 from the left we derive

$$x_0y - yx_1 = -\beta x_0 y^2 x_1, \tag{29}$$

so that $x_0y = (I-\beta x_0y)yx_1$. From the definitions of x_0 and y it follows immediately that $||\beta x_0 y|| \le |\alpha|^{-1} < 1$. Therefore, we have $(I-\beta x_0y)^{-1} = \sum_{n=0}^\infty \beta^n (x_0y)^n$ and hence

$$yx_1 = (I-\beta x_0 y)^{-1} x_0 y = \sum_{n=0}^\infty \beta^n (x_0y)^{n+1}.$$

The latter implies that $(x_0y)yx_1 = yx_1(x_0y)$. Inserting this into (29) we obtain (25). □

Theorem 6. *Let $\alpha,\beta\in\mathbb{R}$, $\alpha<-1$ and $\beta\neq 0$. Suppose that x_0, x_1 and y are bounded linear operators on a Hilbert space $\mathcal{H}$ satifying the equations (23)–(25). Assume that* $\ker x_0 = \ker y = \{0\}$ *and define*

$$A := x_0^{-1} + \alpha i I \quad\text{and}\quad B := y^{-1} + \beta i I. \tag{30}$$

Then A and B are self-adjoint operators on $\mathcal{H}$ and there exists a unitary representation U of the group G on $\mathcal{H}$ such that $i\partial U(\mathsf{a}) = A$ and $i\partial U(\mathsf{b}) = B$.

Proof. The basic pattern of the proof is similar to that of Kato's theorem [K2], but the technical details are more complicated. The self-adjointness of A and B follows from Lemma 11.

First we prove by induction on $n\in\mathbb{N}$ that

$$x_0^n y = yx_1^n + \beta i\; y(x_1^n - x_0^n)y. \tag{31}$$

If $n{=}1$, then (31) holds by combining (25) and the first equality of (24). Suppose that (31) is valid for $n\in\mathbb{N}$. Note that $x_0x_1 = x_1x_0$ by (24). Using first the induction hypothesis, then equation (31) in the case $n{=}1$ and finally once more the induction hypothesis, we compute

$$\begin{aligned} x_0^{n+1}y &= x_0(yx_1^n + \beta i\; y(x_1^n - x_0^n)y) = (yx_1 + \beta i\; y(x_1-x_0)y)(x_1^n + \beta i\;(x_1^n - x_0^n)y)\\ &= yx_1^{n+1} + \beta i\; y(x_1{-}x_0)yx_1^n + \beta i\; yx_1(x_1^n - x_0^n)y - \beta^2 y(x_1{-}x_0)(x_1^n - x_0^n)y\\ &= yx_1^{n+1} + \beta i\; y(x_1{-}x_0)(x_0^n y - \beta i\; y(x_1^n - x_0^n)y) + \beta i\; yx_1(x_1^n - x_0^n)y - \beta^2 y(x_1{-}x_0)(x_1^n - x_0^n)y\\ &= yx_1^{n+1} + \beta i\; y((x_1{-}x_0)x_0^n + x_1(x_1^n - x_0^n))y = yx_1^{n+1} + \beta i\; y(x_1^{n+1} - x_0^{n+1})y, \end{aligned}$$

which completes the induction proof of equation (31).

Let $\mathcal{F}_1$ denote the set of all complex λ for which λ and $\lambda+i$ are not real and the identity

$$(A+i-\lambda)^{-n}y = y(A-\lambda)^{-n} + \beta i\; y((A-\lambda)^{-n} - (A+i-\lambda)^{-n})y \tag{32}$$

holds for all $n\in\mathbb{N}$. Suppose that $\lambda_0\in\mathcal{F}_1$. Fix $k\in\mathbb{N}$. Let λ be a complex number such that $|\lambda-\lambda_0|(||(A-\lambda_0)^{-1}|| + ||(A+i-\lambda_0)^{-1}||) < 1$. We multiply equation (32) by $\binom{n-1}{k-1}(\lambda-\lambda_0)^{n-k}$ and sum over $n=k,k{+}1,\dots$. Using the identities

$$(A-\lambda)^{-k} = \sum_{n=k}^\infty \binom{n-1}{k-1}(\lambda-\lambda_0)^{n-k}(A-\lambda_0)^{-n},$$

$$(A+i-\lambda)^{-k}=\sum_{n=k}^{\infty}\binom{n-1}{k-1}(\lambda-\lambda_0)^{n-k}(A+i-\lambda_0)^{-n}$$

we conclude that (32) is satisfied for λ and k. Therefore, $\lambda\in\mathcal{F}_1$ which proves that $\mathcal{F}_1$ is open. Recall that $(A-\alpha i)^{-1}=x_0$ by (30). Combining this fact with (24) we derive

$$(A-\alpha i-i)x_1=(A-\alpha i)x_0(I+ix_1)-ix_1=I$$

and similarly $x_1(A-\alpha i-i)=I$, so that $(A-\alpha i-i)^{-1}=x_1$. Inserting these formulas for x_0 and x_1 into (31) we obtain equation (32) for $\lambda=i+\alpha i$. That is, $i+\alpha i\in\mathcal{F}_1$. Because $\mathcal{F}_1$ is open as just shown, the connected component of $i+\alpha i$ in the complement of $\mathbb{R}\cup(\mathbb{R}+i)$ is contained in $\mathcal{F}_1$. Since $\alpha<-1$ by assumption, (32) holds for all λ of the lower half-plane.

Multiplying (32) by $(-\lambda)^n$ and setting $\lambda=-n\gamma^{-1}i$ with $\gamma>0$ and $n\in\mathbb{N}$, we obtain

$$(I-\gamma n^{-1}i(A+i))^{-n}y=y(I-\gamma n^{-1}iA)^{-n}+\beta i\, y((I-\gamma n^{-1}iA)^{-n}-(I-\gamma n^{-1}i(A+i))^{-n})y \tag{33}$$

We now need the following fact (see e.g. [HPh], p. 362 or [K1], p. 479): If C is the infinitesimal generator of a contraction semigroup $\{e^{\gamma C};\gamma\geq 0\}$, then we have

$$e^{\gamma C}=s\!-\!\lim_{n\to\infty}(I-\gamma n^{-1}C)^{-n}. \tag{34}$$

Applying this formula to the generators iA and $i(A+i)$ of contraction semigroups, it follows from (33) that

$$e^{\gamma i(A+i)}y=ye^{\gamma iA}+\beta i\, y(e^{\gamma iA}-e^{\gamma i(A+i)})y$$

for all $\gamma>0$. Because $(B-\beta i)^{-1}=y$ by (30), the latter yields

$$e^{i\gamma A}y=ye^{i\gamma A}(e^{\gamma}(I+\beta i\, y)-\beta i\, y)=ye^{i\gamma A}(e^{\gamma}B-\beta i)y.$$

Hence we have

$$e^{i\gamma A}(e^{\gamma}B-\beta i)^{-1}=ye^{i\gamma A}=(B-\beta i)^{-1}e^{i\gamma A}$$

which in turn implies that

$$e^{i\gamma A}(e^{\gamma}B-\beta i)^{-n}=(B-\beta i)^{-n}e^{i\gamma A} \tag{35}$$

for all $n\in\mathbb{N}$ and $\gamma>0$.

Now we fix $\gamma>0$ and consider the set $\mathcal{F}_2$ of all $\lambda\in\mathbb{C}\backslash\mathbb{R}$ for which

$$e^{i\gamma A}(e^{\gamma}B-\mu)^{-n}=(B-\mu)^{-n}e^{i\gamma A} \tag{36}$$

is satisfied for all $n\in\mathbb{N}$. Arguing as in the paragraph before last, with A and $A+i$ replaced by B and $e^{\gamma}B$, we conclude that $\mathcal{F}_2$ is open. Since $\beta i\in\mathcal{F}_2$ by (35), $\mathcal{F}_2$ contains the lower half-plane when $\beta<0$ resp. the upper half-plane when $\beta>0$. Let us first assume that $\beta<0$. Then (36) is valid for all μ such that Im $\mu<0$.

Proceeding as above, we multiply equation (36) by $(-\mu)^n$ and set $\mu=-n\delta^{-1}i$ with $\delta>0$ and $n\in\mathbb{N}$. Letting $n\to\infty$ by using formula (34) we obtain

$$e^{i\gamma A}e^{i\delta e^{\gamma}B}=e^{i\delta B}e^{i\gamma A}. \tag{37}$$

Up to now equation (37) has been proved only for $\gamma>0$ and $\delta>0$. We now show that (37) holds for arbitary real numbers γ and δ. First we note that (37) is trivially fulfilled if $\gamma=0$ or $\delta=0$. Applying the involution to (37) and multiplying the corresponding equation by $e^{i\gamma A}$ from the left and from the right we get $e^{i\gamma A}e^{-i\delta e^{\gamma}B}=e^{-i\delta B}e^{i\gamma A}$. This shows that (37) is valid for all $\gamma\geq 0$ and $\delta\in\mathbb{R}$. Applying the involution to (37), with δ replaced by real η, and multiplying then by $e^{i\eta e^{\gamma}B}$ from the left and by $e^{i\eta B}$ from the right we derive $e^{-i\gamma A}e^{i\eta B}=e^{i\eta e^{\gamma}B}e^{-i\gamma A}$. Setting $\delta=\eta e^{\gamma}$ the latter yields $e^{-i\gamma A}e^{i\delta e^{-\gamma}B}=e^{i\delta B}e^{-i\gamma A}$ which means that (37) holds for $\gamma\leq 0$ and $\delta\in\mathbb{R}$. Thus, equation (37) is satisfied for all reals γ,δ.

The case when $\beta>0$ is treated a in similar manner replacing $\delta>0$ by $\delta<0$ in the preceding.

For $(e^{\gamma},\delta)\equiv\exp\,\delta\mathsf{b}\,\exp\,\gamma\mathsf{a}\in G$ we define $U((e^{\gamma},\delta))=e^{-i\delta B}e^{-i\gamma A}$. A straightforward computation based on equation (37) shows that U is a homomorphism of G into the unitary group of $\mathcal{H}$. Hence U is a unitary representation of G on $\mathcal{H}$. Clearly, $i\partial U(\mathsf{a})=A$ and $i\partial U(\mathsf{b})=B$. $\square$

As a byproduct of the preceding considerations the next theorem gives an integrability criterion for Hilbert space representations of the Lie algebra $\mathfrak{g}$. Here the density condition (39) is the crucial assumption for the integrability of the representation. Note that it is not sufficent that A and B are selfadjoint operators satisfying relation (38) on a common core.

Theorem 7. *(i) Suppose that U is a unitary representation of G. Let α, β be fixed real numbers such that $|\alpha| > 1$ and $\beta \neq 0$. Let $A = i\partial U(\mathsf{a})$, $B = i\partial U(\mathsf{b})$ and $\mathcal{D} = (A-\alpha i)^{-1}(B-\beta i)^{-1}\mathcal{H}(U)$. Then $\mathcal{D}$ is dense in $\mathcal{H}(U) = (B-\beta i)(A-\alpha i)\mathcal{D}$ and we have*

$$AB\varphi - BA\varphi = iB\varphi \quad \text{for} \quad \varphi \in \mathcal{D}. \tag{38}$$

(ii) Suppose that A and B are self-adjoint operators on a Hilbert space $\mathcal{H}$. Let $\alpha, \beta \in \mathbb{R}$, $\alpha < -1$ and $\beta \neq 0$. Assume that there is a linear subspace $\mathcal{D} \subseteq \mathcal{D}(AB) \cap \mathcal{D}(BA)$ of $\mathcal{H}$ such that (38) holds and that

$$(B-\beta i)(A-\alpha i)\mathcal{D} \ \text{ or } \ (A-(\alpha+1)i)(B-\beta i)\mathcal{D} \ \text{ is dense in } \ \mathcal{H}. \tag{39}$$

Then there exists a unitary representation U of G such that $A = i\partial U(\mathsf{a})$ and $B = i\partial U(\mathsf{b})$.

Proof. We retain the notations $x_0 = (A-\alpha i)^{-1}$, $x_1 = (A-(\alpha+1)i)^{-1}$ and $y = (B-\beta i)^{-1}$.

(i): Recall that equation (25) is satisfied by Proposition 6 and (30) holds by definition. Obviously, $\ker\,(x_0y)^* = \{0\}$, so $\mathcal{D} = x_0y\mathcal{H}$ is dense in $\mathcal{H}(U)$.

From (25) and (30) it follows that $\mathcal{D} \subseteq \mathcal{D}(AB) \cap \mathcal{D}(BA)$. If $\psi \in \mathcal{H}$, then $\varphi := x_0y\psi = yx_1(I-\beta x_0y)\psi \in \mathcal{D}$ by (25). To prove that equation (38) is valid we compute

$$\begin{aligned} AB\varphi - BA\varphi - iB\varphi &= (A-(\alpha+1)i))(B-\beta i)\varphi - (B-\beta i)(A-\alpha i)\varphi + \beta\varphi \\ &= (A-(\alpha+1)i)(B-\beta i)yx_1(I-\beta x_0y)\psi - (B-\beta i)(A-\alpha i)x_0y\psi + \beta x_0y\psi \\ &= (I-\beta x_0y)\psi - \psi + \beta x_0y\psi = 0. \end{aligned}$$

(ii): Assume that $\mathcal{D}_1 = (B-\beta i)(A-\alpha i)\mathcal{D}$ is dense in $\mathcal{H}$. The case when $(A-(\alpha+1)i)(B-\beta i)\mathcal{D}$ is dense is treated in a similar manner. Let $\varphi \in \mathcal{D}_1$. Then $\varphi = (B-\beta i)(A-\alpha i)\psi$ for some $\psi \in \mathcal{D}$. By (38) we have $\varphi = (A-(\alpha+1)i)(B-\beta i)\psi + \beta\psi$, so that $x_0y\varphi = \psi$ and $yx_1\varphi = \psi + \beta yx_1\psi$ which in turn yields that $x_0y\varphi = yx_1\varphi - \beta yx_1x_0y\varphi = yx_1\varphi - \beta yx_0x_1y\varphi$. Since $\mathcal{D}_1$ is dense in $\mathcal{H}$, we have $x_0y = yx_1 - \beta yx_0x_1y$ on $\mathcal{H}$, that is, (25) holds. Since equations (23) and (24) follow at once from the definitions of x_0, x_1 and y, Theorem 6 applies and gives the assertion. □

8. Application: A Strict Positivstellensatz for the Enveloping Algebra of the $ax+b$-Group

In this section $\mathcal{A}$ is the complex universal enveloping algebra $\mathcal{E}(\mathfrak{g})$ of the Lie algebra $\mathfrak{g}$ of the affine group of the real line. Setting $a := i\mathsf{a}$ and $b := i\mathsf{b}$, $\mathcal{A}$ becomes the unital $*$-algebra with two hermitian generators a and b and defining relation

$$ab - ba = ib. \tag{40}$$

Let us fix two reals α and β such that $\alpha < -1$, $\beta \neq 0$ and α is not an integer and set

$$\mathcal{S}_g = \{s = b - \beta i,\ s_n = a - (\alpha+n)i;\ n \in \mathbb{Z}\},\ \mathcal{S}_G = \mathcal{S}_g \cup \mathcal{S}_g^*,\ \mathfrak{X}_G = \mathcal{S}_G^{-1},\ \mathcal{A}_G = \{a,\ b\}.$$

Using (40) we obtain $s_{n+1}b = bs_n$, $s_{n-1}^*b = bs_n^*$ for $n \in \mathbb{Z}$, $s^2a = (s(a-i)+\beta)s$ and $(s^*)^2a = (s^*(a-i)-\beta)s^*$. From these formulas it follows that the unital monoid $\mathcal{S}$ generated by the set $\mathcal{S}_G$ is a $*$-invariant left Ore set, so we can assume that $\mathcal{S} = \mathcal{S}_O$.

The $*$-subalgebra $\mathfrak{X}$ of $\mathcal{A}\mathcal{S}_O^{-1}$ is the unital algebra generated by the elements $y{:=}s^{-1}$ and $x_n{:=}s_n^{-1}$, where $n \in \mathbb{Z}$, and their adjoints. In the $*$-algebra $\mathfrak{X}$ we have the following relations:

$$x_n - x_n^* = 2(\alpha+n)i\, x_n^*x_n = 2(\alpha+n)i\, x_nx_n^*,\ \ y - y^* = 2\beta i\ y^*y = 2\beta i yy^*, \tag{41}$$

$$x_n - x_k = (n-k)ix_nx_k = (n-k)ix_kx_k,\ \ x_n - x_k^* = (2\alpha+k+n)i\, x_nx_k^* = (2\alpha+k+n)i\, x_k^*x_n, \tag{42}$$

$$x_ny - yx_{n+1} = -\beta yx_{n+1}x_ny = -\beta x_ny^2x_{n+1},\ \ x_ny^* - y^*x_{n+1} = \beta y^*x_{n+1}x_ny^* = \beta x_n(y^*)^2x_{n+1}. \tag{43}$$

Lemma 12. *Conditions (O), (IA) and (AB) are satisfied.*

Proof. The proof is similar to that of Lemma 10. As a sample, we verify (IA). Combining relations (42) and (43) we obtain $x_n x_k = x_k x_n$, $x_n x_k^* = x_k^* x_n$,

$$\begin{aligned} x_n y &= y x_{n+1}(1-\beta x_n y) = (1-\beta x_n y) y (1+i x_{n+1}) x_n, \\ x_n y^* &= y^* x_{n+1}(1+\beta x_n y^*) = (1+\beta x_n y^*) y^* (1+i x_{n+1}) x_n, \\ x_n^* y &= y x_{n-1}^*(1-\beta x_n^* y) = (1-\beta x_n^* y) y (1+i x_{n-1}^*) x_n^*, \\ x_n^* y^* &= y^* x_{n-1}^*(1+\beta x_n^* y^*) = (1+\beta x_n^* y^*) y^* (1+i x_{n-1}^*) x_n^* \end{aligned}$$

for $n, k \in \mathbb{Z}$. From these equations and their adjoints we conclude that (IA) is fulfilled. □

Proposition 7. *For any $\mathcal{S}^{-1}$-torsionfree $*$-representation ρ of the $*$-algebra $\mathfrak{X}$ there exists a unique unitary representation U of the group G such that $\pi_\rho = dU$. The representation ρ is irreducible if and only if U is irreducible.*

Proof. Since the relations (23)–(25) are contained in (41)–(43), Theorem 6 applies. Hence there exists a unitary representation U of G such that $i\partial U(\mathsf{a}) = A$ and $i\partial U(\mathsf{b}) = B$. As in the proof of Proposition 5 it follows that $\pi_\rho(a)\varphi = A\varphi$ and $\pi_\rho(b)\varphi = B\varphi$ for $\varphi \in \mathcal{D}(\pi_\rho)$ and that $\mathcal{D}(\pi_\rho)$ is the intersection of ranges of all finite products of operators $(A - i(\alpha+n))^{-1} = \rho(x_n) = \rho(s_n^{-1})$, $(B - \beta i)^{-1} = \rho(y) = \rho(s_2^{-1})$ and their adjoints. The latter set is obviously the intersection of domains of all finite products of A and B. Hence $\mathcal{D}(\pi_\rho)$ is equal to the domain $\mathcal{D}^\infty(U)$ (see e.g. [S1], Theorem 10.1.9) of dU. Since $dU(a)\psi = i\partial U(\mathsf{a})\psi = A\psi$ and $dU(b)\psi = i\partial U(\mathsf{b})\psi = B\psi$ for $\psi \in \mathcal{D}^\infty(U)$, we conclude that $\pi_\rho = dU$.

As stated in Theorem 1, ρ is irreducible if and only if π_ρ is so. But $dU = \pi_\rho$ is known to be irreducible if and only if the unitary representation U is irreducible ([S1], 10.2.18). □

From Proposition 6 it follows easily the converse of Proposition 7 is also true (that is, any $*$-representation dU of $\mathcal{A}$ is equal to π_ρ for some torsionfree $*$-representation ρ of $\mathfrak{X}$), but we will need this result in what follows.

Because $\{a^n b^k; k, n \in \mathbb{N}_0\}$ and $\{b^n a^k; k, n \in \mathbb{N}_0\}$ are bases of the vector space $\mathcal{A}$ by the Poincare-Birkhoff-Witt theorem, each nonzero element $c \in \mathcal{A}$ can be written as

$$c = \sum_{j=0}^{d_1} \sum_{l=0}^{d_2} \gamma_{jl} a^j b^l = \sum_{n=0}^{d_2} f_n(a) b^n = \sum_{k=0}^{d_1} g_k(b) a^n. \tag{44}$$

Here $\gamma_{jl} \in \mathbb{C}$ and $f_n(a)$ and $g_k(b)$ are complex polynomials uniquely determined by c. We define $d(c) = (d_1, d_2)$ if there are numbers $j_0, l_0 \in \mathbb{N}_0$ such that $\gamma_{d_1,l_0} \neq 0$ and $\gamma_{j_0,d_2} \neq 0$. Then d is a multi-degree map on the $*$-algebra $\mathcal{A}$. It is easily checked that conditions $(A3)$–$(A5)$ are valid.

Theorem 8. *Supoose that $c = c^*$ is a nonzero element of the enveloping algebra $\mathcal{A} = \mathcal{E}(\mathfrak{g})$ with multi-degree $d(c) = (2m_1, 2m_2)$, where $m_1, m_2 \in \mathbb{N}_0$, satisfying the following assumptions:*
(I) For each irreducible unitary representation U of G there exists a bounded self-adjoint operator $T_U > 0$ on $\mathcal{H}(U)$ such that $dU(c) \geq T_U$.
(II) $\gamma_{2m_1,2m_2} \neq 0$ and the polynomials $f_{2m_2}(\cdot + m_2 i)$ and g_{2m_1} are positive on the real line.
Then there exists an element $s \in \mathcal{S}$ such that $s^* c s \in \sum \mathcal{A}^2$.

Proof. Since the proof follows a similar pattern as the proof of Theorem 5, we sketch only the necessary modifications. Setting $t := s^{m_2} s_0^{m_1}$, the element $z := t^{-1} c (t^*)^{-1} = x_0^{m_1} y^{m_2} c (y^*)^{m_1} (x_0^*)^{m_1}$ belongs to $\mathfrak{X}$ by Lemma 7(ii).

The assertion follows from Theorem 3. It remains to prove that assumptions (i) and (ii) therein are satisfied. Assumption (i) is a consequence of assumption (I) combined with Proposition 7. To verify assumption (ii) we first note that all ideals $\mathcal{J}_{s_n}$ coincide by relation (42), so it suffices to show that assumption (II) implies that $\rho_s(z) > 0$ and $\rho_{s_0}(z) > 0$.

Let us begin with $\rho_s(z)$. Note that that we have $f_{2m_2}(a) b^{m_2} = b^{m_2} f_{2m_2}(a + m_2 i)$ by the commutation relation (40). Further, we have $yb = 1 + \beta i y$ and $by^* = 1 - \beta i y^*$. Using these facts and arguing as in the proof of Theorem 5 it follows that

$$\rho_s(z) = \rho_s(x_0^{m_1} y^{m_2} f_{2m_2}(a) b^{2m_2} (y^*)^{m_2} (x_0^*)^{m_1}) = \rho_s(x_0^{m_1} f_{2m_2}(a + m_2 i)(x_0^*)^{m_1}). \tag{45}$$

Now we turn to $\rho_{s_0}(z)$. From (42) and (43) we have $x_0y - yx_0 = (i - \beta y - \beta i x_1 y)x_1$. As in the proof of Theorem 5 we therefore obtain

$$\rho_{s_0}(z) = \rho_{s_0}(x_0^{m_1} y^{m_2} g_{2m_1}(b) a^{2m_1} (y^*)^{m_2} (x_0^*)^{m_1}) = \rho_{s_0}(y^{m_2} x_0^{m_1} g_{2m_1}(b) a^{2m_1} (x_0^*)^{m_1} (y^*)^{m_2})$$

Let $g_{2m_1}(b) = \sum_{l=0}^{2m_2} \gamma_l b^l$. From (40) it follows that $g_{2m_2}(b)a^{m_1} = \sum_{l=0}^{2m_2} \gamma_l (a - li)^{m_1} b^l$. Moreover, $xa = 1 + \alpha i x$. Using these relation we derive

$$\rho_{s_1}(z) = \rho_{s_0}(y^{m_2} x_0^{m_1} g_{2m_1}(b) a^{2m_1} (x_0^*)^{m_1} (y^*)^{m_2}) = \rho_{s_1}(y^{m_2} g_{2m_1}(b)(y^*)^{m_2}) \tag{46}$$

Having (45) and (46) a similar reasoning as in the last part of the proof of Theorem 5 shows that assumption (II) implies that $\rho_s(z) > 0$ and $\rho_{s_0}(z) > 0$. □

Remark 6. According to a classical result due to Gelfand and Naimark [GN], the set of equivalence classes of irreducible unitary representations of the group G consists of two infinite-dimensional representations $U_\pm$ and of a family U_γ, $\gamma \in \mathbb{R}$, of one-dimensional representations. The associated infinitesimal representations $dU_\pm$ act on the domain

$$\mathcal{D}^\infty(U_\pm) = \{f \in C^\infty(\mathbb{R}) : e^{nx} f^{(m)}(x) \in L^2(\mathbb{R}) \text{ for all } n, m \in \mathbb{N}_0\}$$

of the Hilbert space $L^2(\mathbb{R})$ by $dU_\pm(a)f = if'$ and $dU_\pm(b)f = \pm e^x f(x)$. For $\gamma \in \mathbb{R}$ we have $dU_\gamma(a) = \gamma$ and $dU_\gamma(b) = 0$. Inserting these expressions into (44) leads to a more explicit form of assumption (I) of Theorem 8. That is, (I) is equivalent to the requirements $f_0 > 0$ on $\mathbb{R}$ and

$$dU_\pm(c) = \sum_{k=0}^{2m_1} g_k(\pm e^x) i^k \Big(\frac{d}{dx}\Big)^k \geq T_\pm \text{ on } \mathcal{D}^\infty(U_\pm)$$

for some bounded selfadjoint opertors $T_\pm$ on $L^2(\mathbb{R})$ satisfying $T_\pm > 0$.

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